These three exercises generalize Section VI.3 in the textbook and fill in some details.

1. Let $u(x, y)$ be a subharmonic function in a vertical strip $\{ (x, y) \in \mathbb{R}^2 : a < x < b, y \in \mathbb{R} \}$. Suppose that u is not identically equal to $-\infty$ and that u is bounded above.

   Let $m(x)$ denote $\sup\{ u(x, y) : y \in \mathbb{R} \}$ (the supremum of the values of u on a vertical line). The hypothesis implies that the quantity $m(x)$ is a finite real number for each x.

   Your task is to show that $m(x)$ is a convex function on the interval (a, b). In other words, if x_1 and x_2 and x are real numbers such that $a < x_1 < x < x_2 < b$, and a real number t between 0 and 1 is defined by the property that $x = tx_1 + (1-t)x_2$, then

   $$m(x) \le tm(x_1) + (1-t)m(x_2).$$

   An equivalent statement is that

   $$m(x) \le \frac{x_2 - x}{x_2 - x_1} m(x_1) + \frac{x - x_1}{x_2 - x_1} m(x_2) \qquad \text{when} \quad x_1 < x < x_2.$$

   Suggestion: Use as a comparison harmonic function a first-degree polynomial (in x) plus $\varepsilon \operatorname{Re} \cos \dfrac{x + iy - a}{b - a}$. Cut the strip off at a large value of $|y|$ to get a bounded region on which to apply the maximum principle. Then let ε go to 0.

2. *Hadamard's three-lines theorem for holomorphic functions*:

   Let f be a bounded holomorphic function in a vertical strip $\{ x + iy \in \mathbb{C} : a < x < b \}$, and let $M(x)$ denote $\sup\{ |f(x + iy)| : y \in \mathbb{R} \}$. Show that if $a < x_1 < x_2 < b$, then

   $$M(tx_1 + (1-t)x_2) \le M(x_1)^t M(x_2)^{1-t} \qquad \text{when} \qquad 0 < t < 1.$$

   This property of the function M is called *logarithmic convexity*.

   Suggestion: If f is holomorphic, then $\log |f|$ is subharmonic. Apply part 1.

3. *Hadamard's three-circles theorem for holomorphic functions*:

   Let f be a holomorphic function in an annulus $\{ z \in \mathbb{C} : a < |z| < b \}$, and let $M(r)$ denote $\max\{ |f(z)| : |z| = r \}$. Show that if $a < r_1 < r_2 < b$, then

   $$M(r_1^t r_2^{1-t}) \le M(r_1)^t M(r_2)^{1-t} \qquad \text{when} \qquad 0 < t < 1.$$

   An equivalent statement is that if $a < r_1 < r < r_2 < b$, and $t = \dfrac{\log(r_2/r)}{\log(r_2/r_1)}$, then

   $$M(r) \le M(r_1)^t M(r_2)^{1-t}.$$

   Suggestion: If $f(z)$ is holomorphic in an annulus, then $f(e^z)$ is holomorphic in a vertical strip. Apply part 2.