Final Examination

Part A

State *three* of the following theorems from the course.

1. The Riemann mapping theorem.
2. The Weierstrass factorization theorem for entire functions.
3. Montel's theorem about locally bounded families of holomorphic functions.
4. Runge's theorem about polynomial approximation.
5. Picard's great theorem about essential singularities.

Part B

Choose *three* of the following items. For each item, either construct an example satisfying the stated conditions or prove that no example exists (whichever is appropriate).

6. An analytic function f in the unit disk such that $|\operatorname{Re} f(z) - \operatorname{Im} f(z)|$ is bounded but $|\operatorname{Re} f(z)|$ is unbounded.
7. A meromorphic function on $\mathbb{C}$ having zeros at the odd positive integers, poles at the even positive integers, and no other zeros or poles.
8. A sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials such that $\limsup_{n\to\infty} |p_n(z)|^{1/n}$ is finite when $|z| < 1$ but discontinuous when $z = 0$.
9. A zero-free entire function of order $1/2$.
10. A sequence of harmonic functions mapping the unit disk into $\mathbb{R} \setminus \{0\}$ such that the sequence fails to be normal in the extended sense.