Final Examination

Part A

State *three* of the following theorems from the course.

1. The Riemann mapping theorem.
2. The Weierstrass factorization theorem for entire functions.
3. Montel's theorem about locally bounded families of holomorphic functions.
4. Runge's theorem about polynomial approximation.
5. Picard's great theorem about essential singularities.

Solution. See the textbook or the lecture notes for the statements of the theorems.

Part B

Choose *three* of the following items. For each item, either construct an example satisfying the stated conditions or prove that no example exists (whichever is appropriate).

6. An analytic function f in the unit disk such that $|\operatorname{Re} f(z) - \operatorname{Im} f(z)|$ is bounded but $|\operatorname{Re} f(z)|$ is unbounded.

   Solution. The Riemann mapping theorem provides an analytic bijection from the unit disk to the strip $\{ w \in \mathbb{C} : \operatorname{Re}(w) - \pi < \operatorname{Im}(w) < \operatorname{Re}(w) + \pi \}$. (You can replace the number π by your favorite positive number.) So the required function exists.

   An explicit formula for $f(z)$ is $(1+i)\log\dfrac{1-z}{1+z}$. Indeed, the linear fractional map $\dfrac{1-z}{1+z}$ sends the unit disk to the right-hand half-plane. The principal branch of the logarithm maps the right-hand half-plane to a horizontal strip of width π. The factor of $1+i$ stretches and rotates the strip into the desired one.

   More generally, an analytic function mapping the unit disk onto an unbounded subset of the strip will do. An example of such a function is $(1+i)\log(1-z)$.

   Alternatively, you might map the unit disk onto a half strip. Recall from an exercise in Math 617 that the sine function maps the half-strip $\{ w \in \mathbb{C} : |\operatorname{Re} w| < \pi/2 \text{ and } \operatorname{Im} w > 0 \}$ biholomorphically onto the upper half-plane. And the linear fractional map $i \cdot \dfrac{1-z}{1+z}$ takes the unit disk to the upper half-plane. So $(1+i)\sin^{-1}\left(i \cdot \dfrac{1-z}{1+z}\right)$ is another solution to the problem.

7. A meromorphic function on $\mathbb{C}$ having zeros at the odd positive integers, poles at the even positive integers, and no other zeros or poles.

Solution. The Weierstrass factorization theorem makes it possible to write down an infinite product representing an entire function f_1 having zeros precisely at the odd positive integers and another infinite product representing an entire function f_2 having zeros precisely at the even positive integers. The quotient f_1/f_2 solves the problem.

A more concise solution is available as an application of the knowledge that the reciprocal of the Gamma function is an entire function having zeros precisely at the nonpositive integers. Hence the quotient

$$\frac{\Gamma\left(\dfrac{2-z}{2}\right)}{\Gamma\left(\dfrac{1-z}{2}\right)}$$

solves the problem.

8. A sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials such that $\limsup_{n\to\infty} |p_n(z)|^{1/n}$ is finite when $|z| < 1$ but discontinuous when $z = 0$.

Solution. The simplest solution is to set $p_n(z)$ equal to z (the same function for every n). Since

$$\lim_{n\to\infty} |z|^{1/n} = \begin{cases} 0, & \text{when } z = 0, \\ 1, & \text{when } 0 < |z| < 1, \end{cases}$$

the requirement of the problem is met.

Remark. The example shows that the upper envelope of a family of subharmonic functions can fail to be subharmonic by failing to be upper semicontinuous.

9. A zero-free entire function of order $1/2$.

Solution. No such function exists.

A zero-free entire function has the form e^g for some entire function g. If e^g has finite order, then a special case of Hadamard's factorization theorem implies that the exponent g is a polynomial. The order of e^g is then an integer, namely, the degree of the polynomial g. So the order cannot be equal to $1/2$. See also Theorem XI.3.7 in the textbook.

10. A sequence of harmonic functions mapping the unit disk into $\mathbb{R} \setminus \{0\}$ such that the sequence fails to be normal in the extended sense.

Solution. No such sequence exists.

Indeed, a continuous real-valued function that is never equal to 0 must have a fixed sign (by the intermediate-value theorem from real calculus). Accordingly, there is either a subsequence consisting of positive functions or a subsequence consisting of negative functions. In the second case, changing the sign reduces to the first case.

So there is no loss of generality in supposing that the original sequence consists of positive harmonic functions. You showed in an exercise (submitted on April 2) that the family of positive harmonic functions in the unit disk is normal in the extended sense.

Alternatively, you could argue that since the disk is simply connected, each harmonic function u on the disk is the real part of some analytic function f. Since the range of f is connected and is disjoint from the imaginary axis, the range is a subset of either the right-hand half-plane or the left-hand half-plane. Suppose without loss of generality that the range lies in the right-hand half-plane.

Accordingly, the original sequence of harmonic functions gives rise to an associated sequence of analytic functions having range contained in the right-hand half-plane. You know from a problem on the midterm exam that this family of analytic functions is normal in the extended sense. (You could instead invoke Montel's fundamental normality criterion.) But there is a subtlety to consider here: a sequence of analytic functions might converge to ∞ without the sequence of real parts converging to ∞ (or converging at all).

One way to sidestep this difficulty is to consider not f but instead e^{-f}, which has range contained in the unit disk. The sequence of these exponential functions is bounded, hence normal in the restricted sense: there is a subsequence converging normally to an analytic function g. Taking the absolute value gives a subsequence of functions of the form e^{-u} converging normally to $|g|$.

Since g is a normal limit of nowhere zero analytic functions, either g is nowhere zero or g is identically zero (by Hurwitz's theorem). In the first case, taking the real logarithm shows that the subsequence of harmonic functions converges normally to $-\log|g|$. In the second case, the subsequence of harmonic functions converges normally to $+\infty$.