Math 650-600: Several Complex Variables

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Taylor expansion

If ρ is a real-valued class C^2 function, then

$$\rho(z) = \rho(0) + 2\,\mathrm{Re}\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(0)z_j + \mathrm{Re}\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial z_k}(0)z_j z_k$$
$$+ \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(0)z_j \bar{z}_k + o(|z|^2).$$

Suppose ρ is a defining function for a domain, and 0 is a boundary point. Choose coordinates such that $\mathrm{Re}\, z_n$ is normal at 0. The Taylor expansion reduces (possibly after rescaling) to

$$\rho(z) = \mathrm{Re}\, z_n + \mathrm{Re}\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial z_k}(0)z_j z_k + \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(0)z_j \bar{z}_k + o(|z|^2).$$

Consequences of a negative eigenvalue

Now $z_n \mapsto z_n + \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial z_k}(0)z_j z_k$ is a holomorphic change of variables that further reduces the Taylor expansion to

$$\rho(z) = \mathrm{Re}\, z_n + \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(0)z_j \bar{z}_k + o(|z|^2).$$

Suppose the Levi form at 0 is negative in some complex tangential direction, say z_1. Then $\rho(\lambda, 0, \dots, 0) < 0$ when λ is a non-zero complex number of modulus less than some ϵ.

Thus the analytic disc $\{\, (\lambda, 0, \dots, 0) : |\lambda| < \epsilon \,\}$ is contained in Ω except for the center point, which lies on the boundary. Translating the disc in the direction of decreasing $\mathrm{Re}\, z_n$ moves the disc inside Ω, so the continuity principle is violated.

Translating the disc in the direction of increasing $\mathrm{Re}\, z_n$ gives a way to extend all holomorphic functions across the boundary by using the Cauchy integral.

Exercise on Reinhardt domains

For a complete Reinhardt domain in $\mathbb{C}^2$ with smooth boundary, show that the Levi form is ≥ 0 if and only if the domain is logarithmically convex.

This solves a special case of the Levi problem.

The next exercise solves another special case of the Levi problem.

Exercise on tube domains

An unbounded domain Ω in $\mathbb{C}^n$ is called a tube domain with base G in $\mathbb{R}^n$ if $\Omega = \{ x + iy \in \mathbb{C}^n : x \in G \text{ and } y \in \mathbb{R}^n \}$.

Exercise. Show that a tube domain in $\mathbb{C}^n$ is pseudoconvex if and only if the base G in $\mathbb{R}^n$ is convex.

References: Pierre Lelong, La convexité et les fonctions analytiques de plusieurs variables complexes, *Journal de Mathématiques Pures et Appliquées* (9) **31** (1952) 191–219; H. J. Bremermann, Complex convexity, *Transactions of the American Mathematical Society* **82** (1956) 17–51.

Pierre Lelong

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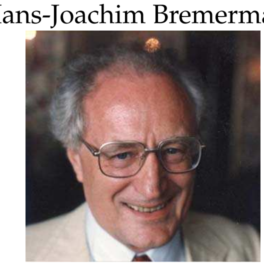

1926–1996

Complete Hartogs domains

A complete Hartogs domain Ω in $\mathbb{C}^{n+1}$ with base G in $\mathbb{C}^n$ is defined by $|z_{n+1}| < e^{-u(z_1,\ldots,z_n)}$ for $(z_1, \ldots, z_n) \in G$, where u is upper semi-continuous.

Theorem. A complete Hartogs domain is pseudoconvex if and only if (a) the base is pseudoconvex and (b) the function u is plurisubharmonic.

Proof. If Ω is pseudoconvex, then there is a plurisubharmonic exhaustion function. Restrict the function to the base to get a plurisubharmonic exhaustion function on G. So (a) holds.

If $c = (0, 0, \ldots, 0, 1)$, then $-\log d_c(z_1, \ldots, z_n, 0) = u(z_1, \ldots, z_n)$. From a previous proof, we know that pseudoconvexity of Ω implies plurisubharmonicity of $-\log d_c$. So (b) holds.

The converse will be done next time.