Math 650-600: Several Complex Variables

Harold P. Boas
boas@tamu.edu

Solvability of the $\overline{\partial}$ problem

Theorem. If Ω is a pseudoconvex domain in $\mathbb{C}^n$, then for every $\overline{\partial}$-closed $(0,1)$-form f with coefficients of class $C^\infty(\Omega)$, there exists a function u of class $C^\infty(\Omega)$ such that $\overline{\partial} u = f$.

We follow Hörmander's L^2 method, a development motivated by ideas of C. B. Morrey, J. J. Kohn, and D. C. Spencer.

References:

- Lars Hörmander, L^2 estimates and existence theorems for the $\overline{\partial}$ operator, *Acta Mathematica* **113** (1965), 89–152.
- Lars Hörmander, *An introduction to complex analysis in several variables*, third edition, North-Holland, Amsterdam, 1990, Chapter IV.
- Lars Hörmander, A history of existence theorems for the Cauchy-Riemann complex in L^2 spaces, *Journal of Geometric Analysis* **13** (2003), no. 2, 329–357.

Hilbert space set-up for $\overline{\partial}$

We work in the Hilbert space $L^2(\Omega, \varphi)$ of functions (or forms) on Ω that are square-integrable with respect to a weight $e^{-\varphi}$. The exponent φ will be a special C^∞ plurisubharmonic exhaustion function for Ω. The inner product $\langle f, g\rangle_\varphi$ on functions equals $\int_\Omega f(z)\overline{g(z)}e^{-\varphi(z)}$; for forms, add the inner products of the corresponding components.

The differential operator $\overline{\partial}$ acts on L^2 in the sense of distributions. The domain of $\overline{\partial}$ is all f for which $\overline{\partial} f$ is in L^2. All smooth, compactly supported functions (or forms) are in the domain of $\overline{\partial}$, which is therefore dense in L^2.

The operator $\overline{\partial}$ is a closed, densely defined (unbounded) operator. There is a closed, densely defined adjoint operator $\overline{\partial}^*$ such that $\langle \overline{\partial} f, g\rangle_\varphi = \langle f, \overline{\partial}^* g\rangle_\varphi$ for $f \in \operatorname{Dom}\overline{\partial}$ and $g \in \operatorname{Dom}\overline{\partial}^*$.

The strategy

First we prove solvability of the $\bar\partial$-equation in suitable weighted L^2 spaces.

Then we show regularity: namely, C^∞ data $\Rightarrow$ C^∞ solution.

The first step (L^2 solvability) depends on the following.

Key estimate. If Ω is a pseudoconvex domain in $\mathbb{C}^n$, then there exist smooth weight functions φ_1, φ_2, and φ_3 such that for every $(0,1)$-form f (that is, $f = \sum_{j=1}^n f_j \, d\bar z_j$) in the intersection of the domains of $\bar\partial$ and $\bar\partial^*$, we have

$$\|f\|_{\varphi_2} \le \|\bar\partial^* f\|_{\varphi_1} + \|\bar\partial f\|_{\varphi_3}.$$

Key estimate $\Rightarrow$ solvability in L^2

The key estimate implies in particular that if f is a $\bar\partial$-closed $(0,1)$-form such that $f \in \operatorname{Dom} \bar\partial^*$, then

$$\|f\|_{\varphi_2} \le \|\bar\partial^* f\|_{\varphi_1}. \tag{1}$$

Suppose $\bar\partial g = 0$; we seek u such that $\bar\partial u = g$.

The map $\bar\partial^* f \mapsto \langle f, g\rangle_{\varphi_2}$ is well-defined on $\bar\partial^*(\operatorname{Dom} \bar\partial^* \cap \ker \bar\partial)$ by (1), linear, and continuous in the topology of the space $L^2(\Omega, \varphi_1)$ because $|\langle f, g\rangle_{\varphi_2}| \le \|g\|_{\varphi_2} \|\bar\partial^* f\|_{\varphi_1}$ by (1). This continuous linear functional extends to all of $L^2(\Omega, \varphi_1)$ by orthogonal projection, and by the Riesz representation theorem there is a function u such that $\langle f, g\rangle_{\varphi_2} = \langle \bar\partial^* f, u\rangle_{\varphi_1}$.

Therefore, by the definition of the adjoint, $\langle f, g\rangle_{\varphi_2} = \langle f, \bar\partial u\rangle_{\varphi_2}$ for all f in the domain of $\bar\partial^*$, which is dense, so $\bar\partial u = g$.

Regularity of the solution

Suppose $\bar\partial u = f = \sum_{j=1}^n f_j \, d\bar z_j \in C^\infty(\Omega)$, that is, $\partial u/\partial \bar z_j \in C^\infty(\Omega)$ for $j = 1, \ldots, n$. We want to show that $u \in C^\infty(\Omega)$.

By Sobolev's lemma, it suffices to show that all partial derivatives of u of all orders are locally square-integrable. In other words, for an arbitrary smooth function ψ with compact support in Ω, we want to show finiteness of

$$\left\| \psi \frac{\partial^{|\alpha|+|\beta|} u}{\partial z^\alpha \partial \bar z^\beta} \right\|^2 = \int_\Omega \psi(z) \frac{\partial^{|\alpha|+|\beta|} u}{\partial z^\alpha \partial \bar z^\beta} \, \overline{\psi(z) \frac{\partial^{|\alpha|+|\beta|} u}{\partial z^\alpha \partial \bar z^\beta}}.$$

Integrating by parts $2|\alpha|$ times eliminates the $\partial/\partial z$ derivatives of u in favor of $\partial/\partial \bar z$ derivatives of u, which are under control by hypothesis.