Math 650-600: Several Complex Variables

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Polynomial approximation

Mergelyan's theorem in the plane. If K is compact and $\mathbb{C} \setminus K$ is connected, then every continuous function on K that is holomorphic in the interior of K can be approximated uniformly on K by holomorphic polynomials.

Exercise. The conclusion of Mergelyan's theorem holds on the bidisc in $\mathbb{C}^2$.

Exercise. The conclusion of Mergelyan's theorem does not hold on the Hartogs triangle in $\mathbb{C}^2$.

The Hartogs phenomenon: version 3

Theorem. Let K be a compact subset of an open set Ω in $\mathbb{C}^n$ with the property that $\Omega \setminus K$ is connected.
If $n \geq 2$, then every holomorphic function on $\Omega \setminus K$ extends holomorphically to Ω.

Corollary. Singular sets of holomorphic functions propagate out to the boundary. So do zero sets of holomorphic functions.

We will prove the theorem by using the solvability of the inhomogeneous $\bar{\partial}$-equation with compact support when $n \geq 2$.

Inhomogeneous Cauchy-Riemann equations

$\bar\partial f = g$, where $g = \sum_{j=1}^{n} g_j \, d\bar z_j$, represents (when $n > 1$) a *system* of partial differential equations

$$\frac{\partial f}{\partial \bar z_j} = g_j, \qquad 1 \le j \le n.$$

When $n > 1$, there is a necessary compatibility condition

$$\frac{\partial g_j}{\partial \bar z_k} = \frac{\partial g_k}{\partial \bar z_j} \qquad \text{for all } j \text{ and } k.$$

This compatibility condition may be written as $\bar\partial g = 0$, since $\bar\partial g = \sum_{j=1}^{n} (\bar\partial g_j) \wedge d\bar z_j$, and the wedge product on differential forms of degree 1 is anti-commutative.

The $\bar\partial$-problem

If $\bar\partial g = 0$, does there exist f such that $\bar\partial f = g$?

The problem is solvable *locally*, but whether a *global* solution exists depends on the domain (when $n \ge 2$).

Solution of the $\bar\partial$-problem in the plane. If g has continuous first partial derivatives on the closure of a bounded domain G in $\mathbb{C}$, then a solution of the equation $\frac{\partial f}{\partial \bar z} = g$ is given by

$$f(z) = \frac{1}{2\pi i} \int_G \frac{g(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar\zeta.$$

We will see that the proof follows from Cauchy's formula with remainder.