Math 650-600: Several Complex Variables

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Announcement

Math Club Meeting

Complex Numbers and Geometry

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Thursday, February 17, 6:30 PM

Blocker 627

Solving $\overline{\partial}$ when $n \geq 2$

Theorem. When $n \geq 2$, if $g = \sum_{j=1}^{n} g_j(z)\, d\overline{z}_j$ satisfies the compatibility condition $\overline{\partial} g = 0$, and if each g_j is a continuously differentiable function having compact support in $\mathbb{C}^n$, then there is a compactly supported continuously differentiable function f such that $\overline{\partial} f = g$.

Exercise. In contrast to the situation when $n \geq 2$, show that in $\mathbb{C}$, if g has compact support but $\int_{\mathbb{C}} g(z)\, dz \wedge d\overline{z} \neq 0$, then no function f such that $\partial f/\partial \overline{z} = g$ can have compact support.

Ehrenpreis's proof of Hartogs's theorem

Suppose h is holomorphic on the connected open set $\Omega \setminus K$ in $\mathbb{C}^n$, where K is compact, and $n \geq 2$. We seek a holomorphic extension of h to all of Ω.

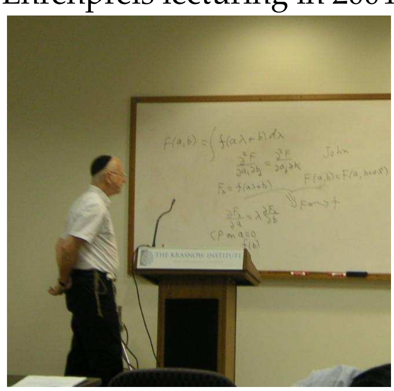
Let φ be a class C^∞ function that is identically equal to 0 in a neighborhood of the compact set K and identically equal to 1 outside a slightly larger neighborhood of K.

The form $h \cdot \bar{\partial}\varphi$ is compactly supported and $\bar{\partial}$-closed on $\mathbb{C}^n$, so there is a compactly supported function f on $\mathbb{C}^n$ such that $\bar{\partial} f = h \cdot \bar{\partial}\varphi$. Off the support of $\bar{\partial}\varphi$, the function f is holomorphic, and f vanishes near infinity, so f is identically equal to 0 on the unbounded component of the complement of the support of $\bar{\partial}\varphi$.

The function $h \cdot \varphi - f$ is holomorphic on Ω and equal to h on an open subset of $\Omega \setminus K$, hence equal to h on all of $\Omega \setminus K$.

Leon Ehrenpreis and his proof

Ehrenpreis lecturing in 2004



The original source for the proof: Leon Ehrenpreis, "A new proof and an extension of Hartog's [sic] theorem", *Bulletin of the American Mathematical Society* **67** (1961) 507–509.

Domains of holomorphy: basic problems

Problem. Characterize domains of holomorphy

- via a geometric property, or
- via an internal property (rather than an external property).

Examples. What are some examples of domains of holomorphy?

Problem. Can one produce more examples of domains of holomorphy via intersections? unions? Cartesian products? other simple constructions?

Analogy: convex domains

External characterization of convexity. A domain Ω in $\mathbb{R}^n$ is convex if the complement of Ω is a union of hyperplanes.

A hyperplane is the set of points $(x_1, \ldots, x_n)$ such that $a_1x_1 + \cdots + a_nx_n = c$, where c is a constant and $(a_1, \ldots, a_n)$ is a vector normal to the hyperplane.

Internal characterization of convexity. A domain Ω in $\mathbb{R}^n$ is convex if for every compact subset K of Ω, the convex hull of K is again a compact subset of Ω.

Function-theoretic definition of the convex hull of K with respect to Ω: the set of points x in Ω that cannot be separated from K by a linear function; in other words, the set of points x such that $L(x) \le \max\{L(y) : y \in K\}$ for every linear function L.

A general notion of convexity

Let $\mathcal{F}$ be a non-empty set of real-valued functions on a domain Ω in $\mathbb{C}^n$. When $K \subset \Omega$, the *$\mathcal{F}$-hull* of K means the set of points of Ω that cannot be separated from K by functions in $\mathcal{F}$; in other words, the set of points z such that $f(z) \le \sup\{f(w) : w \in K\}$ for every function f in $\mathcal{F}$.

The domain Ω is called convex with respect to $\mathcal{F}$ if the $\mathcal{F}$-hull of K is a (relatively) compact subset of Ω whenever K is. A compact set K is called convex with respect to $\mathcal{F}$ if K equals its $\mathcal{F}$-hull.

Main example. When $\mathcal{F}$ is the set of moduli of holomorphic functions on Ω, convexity with respect to $\mathcal{F}$ is called *holomorphic convexity*.

Theorem (to be proved later). A domain Ω is holomorphically convex if and only if Ω is a domain of holomorphy.

Exercises on convexity

Let Ω be a domain in $\mathbb{C}^n$.

Exercise. Let $\mathcal{F}$ be the set of real parts of holomorphic polynomials of degree 1. Show that Ω is convex with respect to $\mathcal{F}$ if and only if Ω is convex in the ordinary geometric sense.

Exercise. Let $\mathcal{F}$ be the set of moduli of holomorphic polynomials of degree 1. Show that Ω is convex with respect to $\mathcal{F}$ if and only if Ω is convex in the ordinary geometric sense.

Corollary. The holomorphically convex hull of a compact set is contained in the convex hull.