# Math 650-600: Several Complex Variables

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## Exercises on convexity

A domain $\Omega$ in $\mathbb{C}^n$ is convex with respect to a set $\mathcal{F}$ of real-valued functions on $\Omega$ if $K \subset\subset \Omega \Rightarrow \widehat{K}_{\mathcal{F}} \subset\subset \Omega$.

**Exercise.** Let $\mathcal{F}$ be the set of moduli of holomorphic polynomials of degree 1. Show that $\Omega$ is convex with respect to $\mathcal{F}$ if and only if $\Omega$ is convex in the ordinary geometric sense.

**Exercise.** Make a Venn diagram showing the relationships among the following concepts:

- convexity
- holomorphic convexity
- polynomial convexity
- weak linear convexity



## Equivalence theorem

Notation: $\Omega$ is a domain in $\mathbb{C}^n$ with boundary $b\Omega$, and $\widehat{K}$ denotes the holomorphically convex hull of $K$ with respect to $\Omega$.

**Theorem.** The following properties are equivalent.

1. $\Omega$ is a domain of holomorphy in the sense that for every point of $b\Omega$ there exists a holomorphic function on $\Omega$ that "does not extend across the boundary near $p$".
2. If $K \subset\subset \Omega$, then $\operatorname{dist}(K, b\Omega) = \operatorname{dist}(\widehat{K}, b\Omega)$.
3. If $K \subset\subset \Omega$, then $\widehat{K} \subset\subset \Omega$; in other words, $\Omega$ is holomorphically convex.
4. $\Omega$ is a domain of holomorphy in the sense that there exists a holomorphic function on $\Omega$ that "does not extend across any part of the boundary".

The implications (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are easy. We will check (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (2).

## Proof that (1) implies (2)

We will prove the contrapositive statement. If $\operatorname{dist}(\widehat{K}, b\Omega)$ is strictly smaller than $\operatorname{dist}(K, b\Omega)$, then there exist a point $p$ in $\widehat{K}$ and a polydisc $D$ of polyradius $r = (r_1, \dots, r_n)$ such that $(K + D) \subset\subset \Omega$ but $(p + D) \cap b\Omega \neq \varnothing$.

If $f$ is holomorphic on $\Omega$, then $|f|$ is bounded by some constant $M$ on $K + D$, and by the Cauchy estimates, each derivative $|f^{(\alpha)}|$ is bounded on $K$ by $M\alpha!/r^\alpha$. Then $|f^{(\alpha)}(p)| \le M\alpha!/r^\alpha$ because $p \in \widehat{K}$.

Therefore $\sum_\alpha \frac{1}{\alpha!} f^{(\alpha)}(p)(z-p)^\alpha$, the Taylor series of $f$, converges in $p + D$, a region that extends across part of the boundary of $\Omega$. Thus all holomorphic functions on $\Omega$ extend across this part of the boundary.

**Remark.** For "dist", we could use any distance function (Euclidean, $\ell^\infty$, $\ell^1$, ...).



## Proof that (3) implies (4)

Exhaust $\Omega$ by a nested sequence of holomorphically convex compact sets $K_j$ and take a sequence of points $p_j$ such that $p_j \in K_{j+1} \setminus K_j$ and for every open ball $B$ that intersects $b\Omega$, every component of $B \cap \Omega$ contains infinitely many of the $p_j$.

Then choose a holomorphic function $f_j$ bounded by $2^{-j}$ on $K_j$ such that $|f_j(p_j)| > j + \sum_{k=1}^{j-1} |f_k(p_j)|$. The series $\sum_j f_j$ converges uniformly on compact sets to a holomorphic function $f$. Since $|f(p_j)| > j - 1$, and the sequence $\{p_j\}$ accumulates everywhere on the boundary, $f$ does not extend across any part of the boundary.

To choose the sequence, start with the rational points enumerated such that each point occurs infinitely often. Let $p_j$ be any point outside $K_j$ and on the line segment joining the $j$th point in the list to a closest point of $b\Omega$.