Math 650-600: Several Complex Variables

Harold P. Boas
boas@tamu.edu

Equivalence theorem from last time

Theorem. The following properties of a domain Ω in $\mathbb{C}^n$ are equivalent.

1. Ω is a domain of holomorphy in the sense that for every point of $b\Omega$ there exists a holomorphic function on Ω that "does not extend across the boundary near p".
2. If $K \subset\subset \Omega$, then $\operatorname{dist}(K, b\Omega) = \operatorname{dist}(\widehat{K}, b\Omega)$.
3. If $K \subset\subset \Omega$, then $\widehat{K} \subset\subset \Omega$; in other words, Ω is holomorphically convex.
4. Ω is a domain of holomorphy in the sense that there exists a holomorphic function on Ω that "does not extend across any part of the boundary".

Exercise

A domain Ω in $\mathbb{C}^n$ is holomorphically convex if and only if for every sequence of points in Ω with no accumulation point in Ω there is a holomorphic function on Ω that is unbounded on the sequence.

Monomial convexity

Theorem. The following properties of a domain Ω in $\mathbb{C}^n$ are equivalent.

1. The domain Ω is convex with respect to the set $\{|z^\alpha| : \alpha_1 \geq 0, \dots, \alpha_n \geq 0\}$ of moduli of monomials.
2. The domain Ω is both a complete Reinhardt domain and a domain of holomorphy.
3. The domain Ω is a complete, logarithmically convex Reinhardt domain.

Thus for domains with a lot of symmetry, we have both a geometric characterization and a function-theoretic characterization of what it means to be a domain of holomorphy.

Proof

(1) $\Rightarrow$ (2). Since the monomials are a subset of the holomorphic functions on Ω, monomial convexity implies holomorphic convexity. So if Ω is convex with respect to the monomials, then Ω is a domain of holomorphy.
Also, the monomial hull of a single point $(w_1, \dots, w_n)$ is the closed polydisc $\{\, (z_1, \dots, z_n) : |z_1| \leq |w_1|, \dots, |z_n| \leq |w_n| \,\}$. So if Ω is convex with respect to the monomials, then Ω is a complete Reinhardt domain.

(2) $\Rightarrow$ (3). If Ω is a domain of holomorphy, then there is a holomorphic function on Ω that cannot be extended to a larger domain. If Ω is a complete Reinhardt domain, then that holomorphic function has a power series expansion that converges in all of Ω. Consequently, Ω is the convergence domain of some power series. We saw previously that convergence domains are logarithmically convex.

Proof (continued)

(3) ⇒ (1). Suppose Ω is a complete, logarithmically convex Reinhardt domain, and let K be a compact subset of Ω. We need to show that the monomial hull of K is a relatively compact subset of Ω.

We can cover K by a finite number of polydiscs whose closures are contained in Ω. Hence we need only consider the case when K is a finite set S of points with no coordinate equal to 0.

If z is in the monomial hull of S, then $|z^\alpha| \le \max_{w\in S} |w^\alpha|$ for all α, and hence $\sum_{j=1}^n \frac{\alpha_j}{|\alpha|} \log |z_j| \le \max_{w\in S} \sum_{j=1}^n \frac{\alpha_j}{|\alpha|} \log |w_j|$. Density of the rational numbers in the real numbers implies that $\sum_{j=1}^n \lambda_j \log |z_j| \le \max_{w\in S} \sum_{j=1}^n \lambda_j \log |w_j|$ for all non-negative real numbers $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 + \cdots + \lambda_n = 1$. The logarithmic convexity of Ω now implies that all such points z lie in a compact subset of Ω.

Set operations

Which of the following classes of domains
(a) convex domains, (b) domains of holomorphy, (c) polynomially convex domains, (d) monomially convex domains, (e) weakly linearly convex domains
are preserved under the following operations?

(i) taking Cartesian products
(ii) taking unions