Math 650-600: Several Complex Variables

Harold P. Boas
boas@tamu.edu

Exercise from last week

A domain Ω in $\mathbb{C}^n$ is holomorphically convex if and only if for every sequence of points in Ω with no accumulation point in Ω there is a holomorphic function on Ω that is unbounded on the sequence.

Set operations

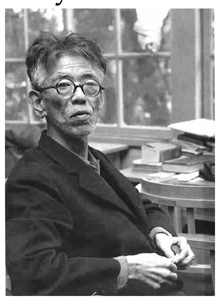
Which of the following classes of domains
(a) convex domains, (b) domains of holomorphy, (c) polynomially convex domains, (d) monomially convex domains, (e) weakly linearly convex domains
are preserved under the following operations?

(i) Cartesian products: all.
(ii) Unions: none.
(iii) Union of an increasing sequence: all, but this fact is non-trivial for the case of domains of holomorphy, in which case it is called the Behnke-Stein theorem.
Also, for the case of polynomial convexity, we need the Oka-Weil theorem (about which more later).
(iv) Connected component of the interior of an intersection: all.

Oka-Weil theorem

Theorem (to be proved later). If K is a polynomially convex compact set in $\mathbb{C}^n$, then every function that is holomorphic on an open neighborhood of K can be approximated uniformly on K by (holomorphic) polynomials.

Kiyoshi Oka



1901–1978

André Weil

1906–1998

Corollary of Oka-Weil

Theorem. Let Ω be a polynomially convex domain in $\mathbb{C}^n$, and let K be a compact subset of Ω. Then the polynomial hull of K with respect to $\mathbb{C}^n$ equals the polynomial hull of K with respect to Ω.

Proof. Write $\widehat{K}_{\mathrm{poly},\mathbb{C}^n} = \widehat{K}_{\mathrm{poly},\Omega} \cup L$, where L is a compact subset of $\mathbb{C}^n \setminus \Omega$.

The function that is equal to 1 in a neighborhood of L and equal to 0 in a neighborhood of $\widehat{K}_{\mathrm{poly},\Omega}$ is holomorphic in a neighborhood of $\widehat{K}_{\mathrm{poly},\mathbb{C}^n}$.

Approximating the function by a polynomial shows that points of L can be separated from K by polynomials. Therefore $L = \varnothing$.

Runge domains

Theorem/definition. A domain of holomorphy Ω is called a *Runge domain* if every holomorphic function on Ω can be approximated uniformly on compact sets by (holomorphic) polynomials. Equivalently, Ω is a polynomially convex domain.

Proof. If Ω is polynomially convex, exhaust Ω by an increasing sequence of polynomially convex compact sets K_j. If f is a holomorphic function on Ω, use Oka-Weil to find polynomials p_j such that $\sup_{K_j} |f - p_j| < 2^{-j}$.

Conversely, suppose Ω is a Runge domain and $K \subset\subset \Omega$.
By hypothesis, a point z can be separated from K by a holomorphic function on Ω if and only if z can be separated from K by a polynomial.
So $\widehat{K}_{\text{poly}} = \widehat{K}_{\text{holo}}$. Consequently $\widehat{K}_{\text{poly}} \subset\subset \Omega$, since Ω is a domain of holomorphy by hypothesis.

Exercises

Are the following subsets of the unit bidisc D^2 domains of holomorphy? polynomially convex?

1. $D^2 \setminus \{\, (x, 0) : 0 \le x \,\}$
2. $D^2 \setminus \{\, (x, 0) : -1 < x < 1 \,\}$
3. $D^2 \setminus \{\, (z_1, 0) : 0 \le \operatorname{Re} z_1 \,\}$
4. $D^2 \setminus \{\, (z_1, 0) : |z_1| < 1 \,\}$