Math 650-600: Several Complex Variables

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Exercises

1. Verify that if u is a plurisubharmonic function of class C^2 on a domain Ω, and if $F : G \to \Omega$ is a holomorphic mapping, then $u \circ F$ is a plurisubharmonic function on G.
The domains G and Ω may be in spaces of different dimensions.

2. By convolving with a mollifier, show that if u is an arbitrary plurisubharmonic function (not necessarily smooth), then u is the limit of a decreasing sequence of smooth plurisubharmonic functions.
More precisely, if u lives on Ω, then there is an exhaustion of Ω by nested subdomains Ω_j, and there is a smooth plurisubharmonic function u_j on Ω_j such that the sequence $u_j \downarrow u$.

More on the continuity principle

Example. $(\mu, \lambda) \mapsto (\mu, \frac{1}{2}(1 + |\mu|)\lambda)$ is a family of analytic discs in λ parametrized by μ (where $|\mu| < 1$ and $|\lambda| \leq 1$). The boundaries of the discs are contained in the "skewered bidisc" $\{ (z_1, z_2) \colon |z_1| < 1, 0 < |z_2| < 1 \}$, but the discs are not wholly contained in that domain.

Continuity principle, version I: If a family of analytic discs is contained in Ω, and if the union of the boundaries of the discs is a relatively compact subset of Ω , then so is the union of the discs. (This is the version of the Kontinuitätssatz stated last time.)

Continuity principle, version II: If $\{D_t\}_{t \in [0,1]}$ is a continuous family of analytic discs in $\mathbb{C}^n$, if the boundaries of the discs are contained in Ω, and if $D_0 \subset \Omega$, then $D_t \subset \Omega$ for every t.

Version I implies version II by connectedness of $[0, 1]$.

Pending theorem on pseudoconvexity

The following properties of a domain Ω in $\mathbb{C}^n$ are equivalent.

1. Ω is convex with respect to plurisubharmonic functions.
2. $z \mapsto -\log \operatorname{dist}(z, b\Omega)$ is plurisubharmonic.
3. A continuous plurisubharmonic exhaustion function exists.

These equivalent properties are called pseudoconvexity.
The Levi problem is to prove that pseudoconvexity is equivalent to holomorphic convexity (to be done later).

Meanwhile, we still need to prove that (1) $\Rightarrow$ (2), which we will accomplish using analytic discs and the continuity principle.

Reduction of the problem

We need to show that for every point a in Ω and every direction b, the function $-\log \operatorname{dist}(a + b\lambda, b\Omega)$ is a subharmonic function of the single complex variable λ.

When c is a unit vector, let $d_c(z)$ denote the distance from z to $b\Omega$ along the complex direction c: namely, $d_c(z) = \sup\{\, r : z + c\lambda \in \Omega \text{ whenever } |\lambda| \le r \,\}$.
Since $-\log \operatorname{dist}(z, b\Omega) = \sup_c(-\log d_c(z))$, it suffices to show that $-\log d_c(a + b\lambda)$ is subharmonic for an arbitrary c.