Math 650-600: Several Complex Variables

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Recap from last time

Continuity principle, version II: If $\{D_t\}_{t\in[0,1]}$ is a continuous family of analytic discs in $\mathbb{C}^n$, if the boundaries of the discs are contained in Ω, and if $D_0 \subset \Omega$, then $D_t \subset \Omega$ for every t.

Pending theorem. The following properties of a domain Ω in $\mathbb{C}^n$ are equivalent.

1. Ω is convex with respect to plurisubharmonic functions.
2. $z \mapsto -\log \operatorname{dist}(z, b\Omega)$ is plurisubharmonic.
3. A continuous plurisubharmonic exhaustion function exists.

To show that (1) ⇒ (2), we reduced to showing, for arbitrary a, b, and c in $\mathbb{C}^n$, that $-\log d_c(a + b\lambda)$ is subharmonic, where $d_c(z)$ denotes the distance to the boundary of Ω in the direction c.

Proof continued

Suppose D is a closed disc in $\mathbb{C}$ such that $a + b\lambda \in \Omega$ when $\lambda \in D$. We need to show that if h is a harmonic function in D such that $-\log d_c(a + b\lambda) \le h(\lambda)$ when λ is in the boundary of D, then the same inequality holds for λ inside D.

Without loss of generality, we may assume that D is the unit disc, $a = 0$, and $h = \operatorname{Re} f$ where f is holomorphic.

Proof concluded

The inequality $-\log d_c(b\lambda) \le \operatorname{Re} f(\lambda)$ may be rewritten as $|e^{-f(\lambda)}| \le d_c(b\lambda)$. Equivalently, $b\lambda + \mu c e^{-f(\lambda)} \in \Omega$ when $|\mu| < 1$. By hypothesis, this holds when $|\lambda| = 1$, and we want to show it holds for $|\lambda| < 1$.

View $b\lambda + \mu c e^{-f(\lambda)}$ as a continuous family of analytic discs in λ parametrized by μ. Let r be an arbitrary positive number such that $r < 1$.

The hypothesis implies that the boundaries of all the discs for which $|\mu| \le r$ lie in (a compact subset of) Ω. When $\mu = 0$, the whole disc is a (compact) subset of Ω by hypothesis.

By version II of the continuity principle, $b\lambda + \mu c e^{-f(\lambda)} \in \Omega$ when $|\mu| \le r$ and $|\lambda| \le 1$. Since r is an arbitrary number less than 1, we are done.

Exercises on pseudoconvexity

Without using the solution of the Levi problem, show that pseudoconvexity is preserved under the following set operations.

1. taking Cartesian products
2. taking increasing unions
3. taking a connected component of the interior of an intersection

Remark. After the solution of the Levi problem, item (2) has the Behnke-Stein theorem as an immediate consequence.