Math 650-600: Several Complex Variables

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Announcement

Math Club movie

The Dalí Dimension

3:00 PM, Wednesday, May 4

Blocker 156

Solving $\overline{\partial}$: the final step

We have solved the $\overline{\partial}$-equation by proving the key estimate

$$\|f\|_{\varphi_2} \le \|\overline{\partial}^* f\|_{\varphi_1} + \|\overline{\partial} f\|_{\varphi_3}$$

for smooth, compactly supported f on a pseudoconvex domain Ω. The $C^\infty(\Omega)$ functions φ_1, φ_2, and φ_3 are given in terms of two auxiliary functions φ and ψ by $\varphi_3 = \varphi$, $\varphi_2 = \varphi - \psi$, and $\varphi_1 = \varphi - 2\psi$. The function φ, constructed last time, is a C^∞ very strongly plurisubharmonic exhaustion function for Ω.

It remains to construct ψ to guarantee density: namely, if f is a $(0,1)$-form in $\operatorname{Dom}\overline{\partial}^* \cap \operatorname{Dom}\overline{\partial}$, then there exists a sequence $\{f^{(j)}\}_{j=1}^\infty$ with coefficients in $C_0^\infty(\Omega)$ such that

$$\|f - f^{(j)}\|_{\varphi_2} + \|\overline{\partial}^* f - \overline{\partial}^* f^{(j)}\|_{\varphi_1} + \|\overline{\partial} f - \overline{\partial} f^{(j)}\|_{\varphi_3} \to 0 \qquad \text{as } j \to \infty.$$

The strategy is to cut f off and mollify.

Choice of ψ

We will use below the condition $\varphi_3 - \varphi_2 = \psi = \varphi_2 - \varphi_1$.

Choose functions η_j in $C_0^\infty(\Omega)$ such that $0 \le \eta_j(z) \le 1$ for all z and $\eta_j(z) = 1$ when $\operatorname{dist}(z, b\Omega) \ge 1/j$. Define $\psi \in C^\infty(\Omega)$ via $e^\psi = 1 + \sum_{j=1}^\infty |\nabla \eta_j|^2$ (locally finite sum).

Density, step 1. If $f \in L^2(\Omega, \varphi_2)$, then $\eta_j f \to f$ in $L^2(\Omega, \varphi_2)$ by the dominated convergence theorem for Lebesgue integrals. If $f \in \operatorname{Dom} \bar\partial$, then similarly $\eta_j \bar\partial f \to \bar\partial f$ in $L^2(\Omega, \varphi_3)$, and if $f \in \operatorname{Dom} \bar\partial^*$, then $\eta_j \bar\partial^* f \to \bar\partial^* f$ in $L^2(\Omega, \varphi_1)$.

Density, step 2. By step 1, showing that $\bar\partial(\eta_j f) \to \bar\partial f$ in $L^2(\Omega, \varphi_3)$ is equivalent to showing that $\|\bar\partial(\eta_j f) - \eta_j \bar\partial f\|_{\varphi_3} \to 0$. The square of this norm is $\le \int_\Omega |\bar\partial \eta_j|^2 \, |f|^2 \, e^{-\varphi_3}$. By the choice of ψ, this integral tends to 0 by dominated convergence since $|\bar\partial \eta_j|^2 |f|^2 \, e^{-\varphi_3} \le e^\psi \, |f|^2 \, e^{-\varphi_3} = |f|^2 \, e^{-\varphi_2}$, and $f \in L^2(\Omega, \varphi_2)$.

More on density

Density, step 3. For compactly supported f, convolve with mollifiers to get f_ϵ with coefficients in $C_0^\infty(\Omega)$ such that $f_\epsilon \to f$ and $\bar\partial f_\epsilon \to \bar\partial f$. Since $\bar\partial$ is a differential operator with *constant coefficients*, it commutes with convolution: $\bar\partial(f_\epsilon) = (\bar\partial f)_\epsilon$.

Density, step 4. Since $\bar\partial^* f = \sum_{j=1}^n e^{\varphi_1} \frac{\partial}{\partial z_j}(e^{-\varphi_2} f_j)$, we have $|\bar\partial^*(\eta_j f) - \eta_j \bar\partial^* f| \le e^{\varphi_1 - \varphi_2} \, |\partial \eta_j| \, |f| \le e^{(\varphi_1 - \varphi_2)/2} \, |f|$ (by the choice of ψ again). Then $|\bar\partial^*(\eta_j f) - \eta_j \bar\partial^* f|^2 \, e^{-\varphi_1} \le |f|^2 \, e^{-\varphi_2}$, so by dominated convergence $\bar\partial^*(\eta_j f) \to \bar\partial^* f$ in $L^2(\Omega, \varphi_1)$.

Density, step 5. Since $e^\psi \, \bar\partial^*$ differs from a constant-coefficient differential operator by a multiplication operator M, we have for compactly supported f that $(e^\psi \, \bar\partial^* + M)(f_\epsilon) = ((e^\psi \, \bar\partial^* + M)f)_\epsilon \to (e^\psi \, \bar\partial^* + M)f$. Since $M(f_\epsilon) \to Mf$, it follows that $e^\psi \, \bar\partial^*(f_\epsilon) \to e^\psi \, \bar\partial^* f$, and hence $\bar\partial^*(f_\epsilon) \to \bar\partial^* f$. We are done.

Summary

Our main result for the semester is the equivalence of the following properties of a domain Ω in $\mathbb{C}^n$.

- Ω is a domain of holomorphy.
- There is a holomorphic function on Ω for which $b\Omega$ is the natural boundary.
- Ω is holomorphically convex.
- $\forall K \subset\subset \Omega$, the holomorphic hull $\widehat{K}$ satisfies $\operatorname{dist}(\widehat{K}, b\Omega) = \operatorname{dist}(K, b\Omega)$.
- $\forall$ sequence in Ω with no accumulation point in Ω,
  $\exists$ a holomorphic function on Ω that is unbounded on the sequence.
- Ω is convex with respect to the plurisubharmonic functions.
- $-\log \operatorname{dist}(z, b\Omega)$ is plurisubharmonic.
- Ω admits a continuous plurisubharmonic exhaustion function.
- Ω admits a C^∞ strongly plurisubharmonic exhaustion function.
- Ω satisfies the continuity principle for families of analytic discs.

Additional equivalences

- In case Ω has class C^2 boundary, the Levi form is non-negative on complex tangent vectors.
- Solvability of the $\bar\partial$-equation on forms of arbitrary degree: When $1 \le q \le n$, for every $\bar\partial$-closed $(0,q)$-form f with coefficients in $C^\infty(\Omega)$, there exists a $(0,q-1)$-form u with coefficients in $C^\infty(\Omega)$ such that $\bar\partial u = f$.

We proved only that pseudoconvexity implies solvability of the $\bar\partial$-equation on $(0,1)$-forms, which is all we needed to solve the Levi problem.