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Lecture notes on several complex variables

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1 Introduction

Although the great Karl Weierstrass (1815–1897) studied multi-variable power series already in the nineteenth century, the modern theory of several complex variables dates to the researches of Friedrich (Fritz) Hartogs (1874–1943) in the first decade of the twentieth century.[1] The so-called Hartogs Phenomenon reveals a dramatic difference between one-dimensional complex analysis and multi-dimensional complex analysis, a fundamental feature that had eluded Weierstrass.

Some aspects of the theory of holomorphic (complex analytic) functions—the maximum principle, for example—are essentially the same in all dimensions. The most interesting parts of the theory of several complex variables are the features that differ from the one-dimensional theory. The one-dimensional theory is illuminated by several complementary points of view: power series, integral representations, partial differential equations, and geometry. The multi-dimensional theory reveals striking new phenomena from each of these points of view. This chapter sketches some of the issues that will be treated in detail later on.

1.1 Power series

A one-variable power series converges inside a certain disc and diverges outside the closure of the disc. The convergence region for a two-dimensional power series, however, can have infinitely many different shapes. For instance, the largest open set in which the double series $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z^n w^m$ converges is the unit bidisc $\{ (z, w) : |z| < 1 \text{ and } |w| < 1 \}$, while the series $\sum_{n=0}^{\infty} z^n w^n$ converges in the unbounded hyperbolic region where $|zw| < 1$.

The theory of one-dimensional power series bifurcates into the theory of entire functions (when the series has infinite radius of convergence) and the theory of functions on the unit disc (when the series has a finite radius of convergence, which can be normalized to the value 1). In higher dimensions, studying power series already leads to function theory on infinitely many different types of domains. A natural question, to be answered later, is to characterize the domains that are convergence domains for multi-variable power series.

Exercise 1. Exhibit a two-variable power series whose convergence domain is the unit ball $\{ (z, w) : |z|^2 + |w|^2 < 1 \}$.

[1] A student of Alfred Pringsheim (1850–1941), Hartogs belonged to the Munich school of mathematicians. Because of their Jewish heritage, both Pringsheim and Hartogs suffered greatly under the Nazi regime in the 1930s. Pringsheim, a wealthy man, managed to buy his way out of Germany into Switzerland, where he died at an advanced age in 1941. The situation for Hartogs, however, grew ever more desperate, and in 1943 he committed suicide rather than face being sent to a death camp.

Hartogs discovered that every function holomorphic in a neighborhood of the boundary of the unit bidisc automatically extends to be holomorphic on the interior of the bidisc; a proof can be carried out by considering one-variable Laurent series on slices. Thus, in dramatic contrast to the situation in one variable, there are domains in $\mathbb{C}^2$ on which all the holomorphic functions extend to a larger domain. A natural question, to be answered later, is to characterize the *domains of holomorphy*, that is, the natural domains of existence of holomorphic functions.

The discovery of Hartogs shows too that holomorphic functions of several variables never have isolated singularities and never have isolated zeroes, in contrast to the one-variable case. Moreover, zeroes (and singularities) must propagate to infinity or to the boundary of the domain where the function is defined.

Exercise 2. Let $p(z, w)$ be a nonconstant polynomial in two variables. Show that the zero set of p cannot be a compact subset of $\mathbb{C}^2$.

1.2 Integral representations

The one-variable Cauchy integral formula for a holomorphic function f on a domain bounded by a simple closed curve C says that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw \qquad \text{for } z \text{ inside } C.$$

A remarkable feature of this formula is that the kernel $(w - z)^{-1}$ is both universal (independent of the curve C) and holomorphic in the free variable z. There is no such formula in higher dimensions! There are integral representations with a holomorphic kernel that depends on the domain, and there is a universal integral representation with a kernel that is not holomorphic. There is a huge literature about constructing and analyzing integral representations for various special types of domains.

1.3 Partial differential equations

The one-dimensional Cauchy–Riemann equations are a pair of real partial differential equations for a pair of functions (the real and imaginary parts of a holomorphic function). In $\mathbb{C}^n$, there are still two functions, but there are $2n$ equations. Thus when $n > 1$, the inhomogeneous Cauchy–Riemann equations form an overdetermined system; hence there is a necessary compatibility condition for solvability of the Cauchy–Riemann equations. This feature is a significant difference from the one-variable theory.

When the inhomogeneous Cauchy–Riemann equations are solvable in $\mathbb{C}^2$ (or in higher dimension), there is (as will be shown later) a solution with compact support in the case of compactly supported data. When $n = 1$, however, it is not always possible to solve the inhomogeneous Cauchy–Riemann equations while maintaining compact support. The Hartogs phenomenon can be interpreted as a manifestation of this dimensional difference.

Exercise 3. Show that if u is the real part of a holomorphic function of two complex variables z_1 $(= x_1+iy_1)$ and z_2 $(= x_2+iy_2)$, then the function u must satisfy the following real second-order partial differential equations:

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial y_1^2}=0, \quad \frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial y_2^2}=0, \quad \frac{\partial^2 u}{\partial x_1 \partial x_2}+\frac{\partial^2 u}{\partial y_1 \partial y_2}=0, \quad \frac{\partial^2 u}{\partial x_1 \partial y_2}=\frac{\partial^2 u}{\partial y_1 \partial x_2}.$$

Thus the real part of a holomorphic function of two variables not only is harmonic in each coordinate but also satisfies additional conditions.

1.4 Geometry

In view of the one-variable Riemann mapping theorem, every bounded simply connected planar domain is biholomorphically equivalent to the unit disc. In higher dimension, there is no such simple topological classification of biholomorphically equivalent domains. Indeed, the unit ball in $\mathbb{C}^2$ and the unit bidisc in $\mathbb{C}^2$ are holomorphically inequivalent domains.

One way to understand intuitively why the situation changes in dimension 2 is to realize that in $\mathbb{C}^2$, there is room for one-dimensional complex analysis to happen in the tangent space to the boundary of a domain. Indeed, the boundary of the bidisc contains pieces of one-dimensional complex affine subspaces, while the boundary of the two-dimensional ball does not contain any such *analytic disc*.

Similarly, the zero set of a (not identically zero) holomorphic function in $\mathbb{C}^2$ is a one-dimensional complex variety, while the zero set of a holomorphic function in $\mathbb{C}^1$ is a zero-dimensional variety (that is, a discrete set of points).

There is a mismatch between the dimension of the domain and the dimension of the range of a multi-variable holomorphic function. One might expect an equidimensional holomorphic *mapping* to be analogous to a one-variable holomorphic function. Here too there are surprises. For instance, there exists a biholomorphic mapping from all of $\mathbb{C}^2$ onto a proper subset of $\mathbb{C}^2$ whose complement has interior points. Such a mapping is called a *Fatou–Bieberbach map*.[2]

[2]The name honors the French mathematician Pierre Fatou (1878–1929) and the German mathematician Ludwig Bieberbach (1886–1982).

2 Power series

Examples in the introduction show that the domain of convergence of a multi-variable power series can have various shapes; in particular, the domain need not be a convex set. Nonetheless, there is a special kind of convexity property that characterizes convergence domains.

Developing the theory requires some notation. The Cartesian product of n copies of the complex numbers $\mathbb{C}$ is denoted by $\mathbb{C}^n$. In contrast to the one-dimensional case, the space $\mathbb{C}^n$ is not an algebra when $n > 1$ (there is no multiplication operation). But the space $\mathbb{C}^n$ is a normed vector space, the usual norm being the Euclidean one: $\|(z_1, \dots, z_n)\| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$. A point $(z_1, \dots, z_n)$ in $\mathbb{C}^n$ is commonly denoted by a single letter z, a vector variable. If α is a point of $\mathbb{C}^n$ all of whose coordinates are non-negative integers, then z^α means the product $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ (as usual, the quantity $z_1^{\alpha_1}$ is interpreted as 1 when z_1 and α_1 are simultaneously equal to 0), the notation $\alpha!$ abbreviates the product $\alpha_1! \cdots \alpha_n!$ (where $0! = 1$), and $|\alpha|$ means $\alpha_1 + \dots + \alpha_n$. In this "multi-index" notation, a multi-variable power series can be written in the form $\sum_\alpha c_\alpha z^\alpha$, an abbreviation for $\sum_{\alpha_1=0}^\infty \cdots \sum_{\alpha_n=0}^\infty c_{\alpha_1,\dots,\alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

There is some awkwardness in talking about convergence of a multi-variable power series $\sum_\alpha c_\alpha z^\alpha$, because the value of a series depends (in general) on the order of summation, and there is no canonical ordering of n-tuples of non-negative integers when $n > 1$.

Exercise 4. Find complex numbers b_α such that the "triangular" sum $\lim_{k\to\infty} \sum_{j=0}^k \sum_{|\alpha|=j} b_\alpha$ and the "square" sum $\lim_{k\to\infty} \sum_{\alpha_1=0}^k \cdots \sum_{\alpha_n=0}^k b_\alpha$ have different finite values.

Accordingly, it is convenient to restrict attention to *absolute* convergence, since the terms of an absolutely convergent series can be reordered arbitrarily without changing the value of the sum (or the convergence of the sum).

2.1 Domain of convergence

The *domain of convergence* of a power series means the interior of the set of points at which the series converges absolutely.[1] For example, the set where the two-variable power series $\sum_{n=1}^\infty z^n w^{n!}$ converges absolutely is the union of three sets: the points (z, w) for which $|w| < 1$ and z is arbitrary, the points $(0, w)$ for arbitrary w, and the points (z, w) for which $|w| = 1$ and $|z| < 1$. The domain of convergence is the first of these three sets; the other two sets contribute no additional interior points.

[1]Usually the domain of convergence is assumed implicitly to be non-void. Ordinarily one would not speak of the domain of convergence of the series $\sum_{n=1}^\infty n!\, z^n w^n$.

Since convergence domains are defined by considering absolute convergence, it is evident that every convergence domain is *multi-circular*: if a point $(z_1, \dots, z_n)$ is in the domain, then so is every point $(\lambda_1 z_1, \dots, \lambda_n z_n)$ such that $|\lambda_1| = \cdots = |\lambda_n| = 1$. Moreover, the comparison test for absolute convergence of series shows that the point $(\lambda_1 z_1, \dots, \lambda_n z_n)$ is still in the convergence domain when $|\lambda_j| \le 1$ for each j. Thus *every convergence domain is a union of polydiscs centered at the origin*. (A polydisc means a Cartesian product of discs, possibly with different radii.)

A multi-circular domain is often called a Reinhardt domain.[2] Such a domain is called *complete* if whenever a point z is in the domain, then the polydisc $\{\, w : |w_1| \le |z_1|, \dots, |w_n| \le |z_n| \,\}$ is in the domain too. The preceding discussion can be rephrased as saying that every convergence domain is a complete Reinhardt domain.

But more is true. If both $\sum_\alpha |c_\alpha z^\alpha|$ and $\sum_\alpha |c_\alpha w^\alpha|$ converge, then Hölder's inequality implies that $\sum_\alpha |c_\alpha||z^\alpha|^t |w^\alpha|^{1-t}$ converges when $0 \le t \le 1$. Indeed, the numbers $1/t$ and $1/(1-t)$ are conjugate indices for Hölder's inequality: the sum of their reciprocals evidently equals 1. In other words, if z and w are two points in a convergence domain, then so is the point obtained by forming in each coordinate the geometric average (with weights t and $1-t$) of the moduli. This property of a Reinhardt domain is called *logarithmic convexity*. Since a convergence domain is complete and multi-circular, the domain is determined by the points with positive real coordinates; replacing the coordinates of each such point by their logarithms produces a convex domain in $\mathbb{R}^n$.

2.2 Characterization of domains of convergence

The following theorem[3] gives a geometric characterization of domains of convergence of power series.

Theorem 1. *A complete Reinhardt domain in $\mathbb{C}^n$ is the domain of convergence of some power series if and only if the domain is logarithmically convex.*

[2] The name honors the German mathematician Karl Reinhardt (1895–1941), who studied such regions. Reinhardt has a place in mathematical history for solving Hilbert's 18th problem in 1928: he found a polyhedron that tiles three-dimensional Euclidean space but is not the fundamental domain of any group of isometries of $\mathbb{R}^3$. In other words, there is no group such that the orbit of the polyhedron under the group covers $\mathbb{R}^3$, yet non-overlapping isometric images of the tile do cover $\mathbb{R}^3$. Later, Heinrich Heesch found a two-dimensional example; Heesch is remembered for pioneering computer methods for attacking the four-color problem.

The date of Reinhardt's death does not mean that he was a war casualty: his obituary says to the contrary that he died after a long illness of unspecified nature.

Reinhardt was a professor in Greifswald, a city in northeastern Germany on the Baltic Sea. The University of Greifswald, founded in 1456, is one of the oldest in Europe. Incidentally, Greifswald is a sister city of Bryan–College Station.

[3] Fritz Hartogs, Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten, *Mathematische Annalen* **62** (1906), no. 1, 1–88. (Hartogs considered domains in $\mathbb{C}^2$.)

Proof. The preceding discussion shows that a convergence domain is necessarily logarithmically convex. What remains to prove is that if D is a logarithmically convex complete Reinhardt domain, then there exists some power series $\sum_\alpha c_\alpha z^\alpha$ whose domain of convergence is D. Suppose initially that the domain D is bounded, for the construction is easier to implement in that case. The idea is to construct a series that can be compared with a geometric series.

Let $N_\alpha(D)$ denote $\sup\{\,|z^\alpha| : z \in D\,\}$, the supremum norm on D of the monomial with exponent α. The hypothesis of boundedness of the domain D guarantees that $N_\alpha(D)$ is finite. The claim is that $\sum_\alpha z^\alpha/N_\alpha(D)$ is the required power series whose domain of convergence is equal to D. What needs to be checked is that for each point w inside D, the series converges absolutely at w, and for each point w outside D, there is no neighborhood of w throughout which the series converges absolutely.

If w is a particular point in the interior of D, then w is in the interior of some open polydisc contained in D, say of polyradius $(r_1, \dots, r_n)$. If λ denotes $\max_{1\le j\le n} |w_j|/r_j$, then $0 < \lambda < 1$, and $|w^\alpha|/N_\alpha(D) \le \lambda^{|\alpha|}$. Therefore the series $\sum_\alpha w^\alpha/N_\alpha(D)$ converges absolutely by comparison with the convergent dominating series $\sum_\alpha \lambda^{|\alpha|}$ (which is a product of convergent geometric series). Thus the first half of the claim is valid.

To check the second half of the claim, it suffices to show that the series $\sum_\alpha |z^\alpha|/N_\alpha(D)$ diverges at an arbitrary point w outside the closure of D whose coordinates are positive real numbers. (Why can one reduce to this case? Since D is multicircular, there is no loss of generality in supposing that the coordinates of w are nonnegative real numbers, and since convergence domains are open sets, there is no loss of generality in supposing that the coordinates of w are strictly positive.) The strategy is to show that infinitely many terms of the series are greater than 1 at w.

The hypothesis that D is logarithmically convex means precisely that the set

$$\{\,(u_1, \dots, u_n) \in \mathbb{R}^n : (e^{u_1}, \dots, e^{u_n}) \in D\,\}, \qquad \text{denoted } \log D,$$

is a convex set in $\mathbb{R}^n$. By assumption, the point $(\log w_1, \dots, \log w_n)$ is a point of $\mathbb{R}^n$ outside the closure of the convex set $\log D$, so this point can be separated from $\log D$ by a hyperplane. In other words, there is a linear function $\ell\colon \mathbb{R}^n \to \mathbb{R}$ whose value at the point $(\log w_1, \dots, \log w_n)$ exceeds the supremum of ℓ over the convex set $\log D$. (In particular, that supremum is finite.) Suppose that $\ell(u_1, \dots, u_n) = \beta_1 u_1 + \cdots + \beta_n u_n$ for every point $(u_1, \dots, u_n)$, where the coefficients β_j are certain real constants.

The hypothesis that D is a complete Reinhardt domain implies that D contains a neighborhood of the origin in $\mathbb{C}^n$, so there is a positive real constant m such that the convex set $\log D$ contains every point u in $\mathbb{R}^n$ for which $\max_{1\le j\le n} u_j \le -m$. Therefore none of the numbers β_j can be negative, for otherwise the function ℓ would take arbitrarily large positive values on $\log D$. The assumption that D is bounded produces a positive real constant M such that $\log D$ is contained in the set of points u in $\mathbb{R}^n$ such that $\max_{1\le j\le n} u_j \le M$. Consequently, if each number β_j is increased by some small positive amount ϵ, then the supremum of ℓ over $\log D$ increases by no more than $nM\epsilon$. Therefore the coefficients of the function ℓ can be perturbed slightly, and ℓ will remain a separating function. Accordingly, there is no loss of generality in assuming that each β_j is a positive rational number. Multiplying by a common denominator shows that the coefficients β_j can be taken to be positive integers.

Exponentiating reveals that $w^\beta > N_\beta(D)$ for the particular multi-index β just determined. (Since the coordinates of w are positive real numbers, no absolute-value signs are needed on the left-hand side of the inequality.) It follows that if k is a positive integer, and $k\beta$ denotes the multi-index $(k\beta_1, \dots, k\beta_n)$, then $w^{k\beta} > N_{k\beta}(D)$. Consequently, the series $\sum_\alpha w^\alpha / N_\alpha(D)$ diverges, for there are infinitely many terms larger than 1. This conclusion completes the proof of the theorem in the special case that the domain D is bounded.

When D is unbounded, let D_r denote the intersection of D with the ball of radius r centered at the origin. Then D_r is a bounded, complete, logarithmically convex Reinhardt domain, and the preceding analysis applies to D_r. It will not work, however, to splice together power series of the type just constructed for an increasing sequence of values of r, for none of these series converges throughout the unbounded domain D.

One way to finish the argument (and to advertise coming attractions) is to apply a famous theorem of H. Behnke and K. Stein (usually called *the* Behnke–Stein theorem), according to which an *increasing* union of domains of holomorphy is again a domain of holomorphy.[4] Section 2.4 will show that a convergence domain for a power series supports some (other) power series that cannot be analytically continued across any boundary point whatsoever. Hence each D_r is a domain of holomorphy, and the Behnke–Stein theorem implies that D is a domain of holomorphy. Thus D supports some holomorphic function that cannot be analytically continued across any boundary point of D. This holomorphic function will be represented by a power series that converges in all of D, and D will be the convergence domain of this power series.

The argument in the preceding paragraph is unsatisfying because, besides being anachronistic and not self-contained, it provides no concrete construction of the required power series. What follows is a nearly concrete argument for the case of an unbounded domain that is based on the same idea as the proof for bounded domains.

Consider the countable set of points outside the closure of D whose n coordinates all are positive rational numbers. (There are such points unless D is the whole space, in which case there is nothing to prove.) Make a redundant list $\{w(j)\}_{j=1}^\infty$ of these points, each point appearing in the list infinitely often. Since the domain D_j is bounded, the first part of the proof provides a multi-index $\beta(j)$ of positive integers such that $w(j)^{\beta(j)} > N_{\beta(j)}(D_j)$. Multiplying this multi-index by a positive integer gives another multi-index with the same property, so it may be assumed that $|\beta(j+1)| > |\beta(j)|$ for every j. The claim is that

$$\sum_{j=1}^\infty \frac{z^{\beta(j)}}{N_{\beta(j)}(D_j)} \tag{2.1}$$

is a power series whose domain of convergence is D.

First of all, the indicated series *is* a power series, since no two of the multi-indices $\beta(j)$ are equal (so there are no common terms in the series that need to be combined). A point z in the

[4] H. Behnke and K. Stein, Konvergente Folgen von Regularitätsbereichen und die Meromorphiekonvexität, *Mathematische Annalen* **116** (1938) 204–216.

interior of D is inside the bounded domain D_k for some value of k, and $N_\alpha(D_j) > N_\alpha(D_k)$ when $j > k$. Therefore the sum of absolute values of terms in the tail of the series (2.1) is dominated by $\sum_\alpha |z^\alpha|/N_\alpha(D_k)$, and the latter series converges for the specified point z inside D_k by the argument in the first part of the proof. Thus the convergence domain of the indicated series is at least as large as D.

On the other hand, if the series were to converge absolutely in some neighborhood of a point outside D, then the series would converge at some point ζ outside the closure of D having positive rational coordinates. Since there are infinitely many values of j for which $w(j) = \zeta$, the series

$$\sum_{j=1}^{\infty} \frac{\zeta^{\beta(j)}}{N_{\beta(j)}(D_j)}$$

has (by construction) infinitely many terms larger than 1, and so diverges. Thus the convergence domain of the constructed series is no larger than D.

In conclusion, every logarithmically convex, complete Reinhardt domain, whether bounded or unbounded, is the domain of convergence of some power series. □

Exercise 5. Every bounded, complete Reinhardt domain in $\mathbb{C}^2$ can be described as the set of points (z_1, z_2) for which

$$|z_1| < r \qquad \text{and} \qquad |z_2| < e^{-\varphi(|z_1|)},$$

where r is some positive real number, and φ is some nondecreasing, real-valued function. Show that such a domain is logarithmically convex if and only if the function $z_1 \mapsto \varphi(|z_1|)$ is subharmonic on the disk where $|z_1| < r$.

2.3 Elementary properties of holomorphic functions

Convergent power series are the local models for holomorphic functions. A reasonable working definition of a holomorphic function of several complex variables is a function (on an open set) that is holomorphic in each variable separately (when the other variables are held fixed) and continuous in all variables jointly.[5]

If D is a polydisc in $\mathbb{C}^n$, say of polyradius $(r_1, \dots, r_n)$, whose closure is contained in the domain of definition of a function f that is holomorphic in this sense, then iterating the one-dimensional Cauchy integral formula shows that

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|w_1|=r_1} \cdots \int_{|w_n|=r_n} \frac{f(w_1, \dots, w_n)}{(w_1 - z_1)\cdots(w_n - z_n)}\, dw_1 \cdots dw_n$$

when the point z with coordinates $(z_1, \dots, z_n)$ is in the interior of the polydisc. (The assumed continuity of f guarantees that this iterated integral makes sense and can be evaluated in any order by Fubini's theorem.)

[5]It is a surprising result of Hartogs that the continuity hypothesis is superfluous. See Section 2.7.

By expanding the Cauchy kernel in a power series, one finds from the iterated Cauchy formula (just as in the one-variable case) that a holomorphic function in a polydisc admits a power series expansion that converges in the (open) polydisc. If the series representation is $\sum_\alpha c_\alpha z^\alpha$, then the coefficient c_α is uniquely determined as $f^{(\alpha)}(0)/\alpha!$, where the symbol $f^{(\alpha)}$ abbreviates the derivative $\partial^{|\alpha|} f/\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}$. Every complete Reinhardt domain is a union of polydiscs, so the uniqueness of the coefficients c_α implies that every holomorphic function in a complete Reinhardt domain admits a power series expansion that converges in the whole Reinhardt domain. Thus holomorphic functions and convergent power series are identical notions in complete Reinhardt domains.

By the same arguments as in the single-variable case, the iterated Cauchy formula suffices to establish standard local properties of holomorphic functions. For example, holomorphic functions are infinitely differentiable, satisfy the Cauchy–Riemann equations in each variable, obey a local maximum principle, and admit local power series expansions.

An identity principle for holomorphic functions of several variables is valid, but the statement is different from the usual one-variable statement. Zeroes of holomorphic functions of more than one variable are never isolated, so requiring an accumulation point of zeroes puts no restriction on the function. The following exercise is a correct version of the identity principle for functions of several variables.

Exercise 6. If f is holomorphic on a connected open set D in $\mathbb{C}^n$, and f is identically equal to 0 on some ball contained in D, then f is identically equal to 0 on D.

The iterated Cauchy integral suffices to show that if a sequence of holomorphic functions converges normally (uniformly on compact sets), then the limit function is holomorphic. Indeed, the conclusion is a local property that can be checked on small polydiscs, and the locally uniform convergence implies that the limit of the iterated Cauchy integrals equals the iterated Cauchy integral of the limit function. On the other hand, the one-variable integral that counts zeroes inside a curve lacks a multi-variable analogue (the zeroes are not isolated), so one needs to check that Hurwitz's theorem generalizes from one variable to several variables.

Exercise 7. Prove a multi-dimensional version of Hurwitz's theorem: On a connected open set, the normal limit of nowhere-zero holomorphic functions is either nowhere zero or identically equal to zero.

2.4 Natural boundaries

Although the one-dimensional power series $\sum_{k=0}^\infty z^k$ has the unit disc as its convergence domain, the function represented by the series, which is $1/(1-z)$, extends holomorphically across most of the boundary of the disc. On the other hand, there exist power series that converge in the unit disc and have the unit circle as "natural boundary," meaning that the function represented by the series does not continue analytically across any boundary point of the disc. (One example is the gap series $\sum_{k=1}^\infty z^{2^k}$, which on a dense of radii has an infinite limit at the boundary.) The following theorem[6] says that in higher dimensions too, every convergence domain (that is, every logarithmically convex, complete Reinhardt domain) is the natural domain of existence of some holomorphic function.

Theorem 2 (Cartan–Thullen)**.** *The domain of convergence of a multi-variable power series is a domain of holomorphy. More precisely, for every domain of convergence there exists some power series that converges in the domain and that is singular at every boundary point.*

The word "singular" does not necessarily mean that the function blows up. To say that a power series is singular at a boundary point of its domain of convergence means that the series does not admit a direct analytic continuation to a neighborhood of the point. A function whose modulus tends to infinity at a boundary point is singular at that point, but so is a function whose modulus tends to zero exponentially fast.

To illustrate some useful techniques, I shall give two proofs of the theorem (different from the original proof). Both proofs are nonconstructive. The arguments show the existence of many noncontinuable series without actually exhibiting a concrete one.

Proof of Theorem 2 using the Baire category theorem. Let D be the domain of convergence (assumed nonvoid) of a power series $\sum_\alpha c_\alpha z^\alpha$. Since the two series $\sum_\alpha c_\alpha z^\alpha$ and $\sum_\alpha |c_\alpha| z^\alpha$ have the same region of absolute convergence, there is no loss of generality in assuming from the outset that the coefficients c_α are non-negative real numbers.

The topology of uniform convergence on compact sets is metrizable, and the space of holomorphic functions on D becomes a complete metric space when provided with this topology. Hence the Baire category theorem is available. The goal is to prove that the holomorphic functions on D that extend holomorphically across some boundary point form a set of first category in this metric space. Consequently, there exist power series that are singular at every boundary point of D; indeed, most power series that converge in D have this property.

A first step toward the goal is a multi-dimensional version of an observation that dates back to the end of the nineteenth century.

Lemma 1 (Multi-dimensional Pringsheim lemma). If a power series $\sum_\alpha c_\alpha z^\alpha$ has real, non-negative coefficients c_α, then the series is singular at every boundary point $(r_1, \dots, r_n)$ of the domain of convergence at which all the coordinates r_j are positive real numbers.

Proof. Seeking a contradiction, suppose that the holomorphic function f represented by the series extends holomorphically to a neighborhood of some boundary point r having positive coordinates. Consider the Taylor series of f about the interior point $\frac{1}{2}r$:

$$f(z) = \sum_\alpha \frac{1}{\alpha!} f^{(\alpha)}(\tfrac{1}{2}r)(z - \tfrac{1}{2}r)^\alpha.$$

[6]Henri Cartan and Peter Thullen, Zur Theorie der Singularitäten der Funktionen mehrerer komplexen Veränderlichen: Regularitäts- und Konvergenzbereiche, *Mathematische Annalen* **106**, (1932) no. 1, 617–647; `doi:10.1007/BF01455905`. See Corollary 1 on page 637 of the cited article.

By the assumption, this series converges when $z = r + \epsilon\mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)$, and ϵ is a sufficiently small positive number. Differentiating the original series shows that

$$f^{(\alpha)}(\tfrac{1}{2}r) = \sum_{\beta \ge \alpha} \frac{\beta!}{(\beta-\alpha)!} c_\beta (\tfrac{1}{2}r)^{\beta-\alpha}.$$

Combining these two expressions shows that the series

$$\sum_\alpha \sum_{\beta \ge \alpha} \binom{\beta}{\alpha} c_\beta (\tfrac{1}{2}r)^{\beta-\alpha} (\tfrac{1}{2}r + \epsilon\mathbf{1})^\alpha$$

converges. Since all the quantities involved are non-negative real numbers, the order of summation can be interchanged without affecting the convergence; the expression then simplifies to the series

$$\sum_\beta c_\beta (r + \epsilon\mathbf{1})^\beta.$$

This series is the original series for f, now seen to be absolutely convergent in a neighborhood of the point r. Hence r is not a boundary point of the domain of convergence. The contradiction shows that f must have been singular at r after all. □

In view of the lemma, the power series $\sum_\alpha c_\alpha z^\alpha$ (now assumed to have non-negative coefficients) is singular at all the boundary points of the domain of convergence having positive real coordinates. (If there are no boundary points, that is, if $D = \mathbb{C}^n$, then there is nothing to prove.) If $(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$ is an arbitrary boundary point having all coordinates non-zero, then the power series $\sum_\alpha c_\alpha e^{-i(\alpha_1\theta_1 + \dots + \alpha_n\theta_n)} z^\alpha$ is singular at this boundary point. In other words, for every boundary point having non-zero coordinates, there exists some power series that converges in D but is singular at that boundary point.

Now choose a countable dense subset $\{p_j\}_{j=1}^\infty$ of the boundary of D consisting of points with non-zero coordinates. For arbitrary natural numbers j and k, the space of holomorphic functions on $D \cup B(p_j, 1/k)$ embeds continuously into the space of holomorphic functions on D via the restriction map. The image of the embedding is not the whole space, for the preceding discussion produced a power series that does not extend into $B(p_j, 1/k)$. By a corollary of the Baire category theorem (dating back to Banach's famous book[7]), the image of the embedding must be of first category (the cited theorem says that if the image were of second category, then it would be the whole space, which it is not). Thus the set of power series on D that extend some distance across some boundary point is a countable union of sets of first category, hence itself a set of first category. Accordingly, most power series that converge in D have the boundary of D as natural boundary. □

[7] Stefan Banach, *Théorie des opérations linéaires*, 1932, second edition 1978, currently available through AMS Chelsea Publishing; English translation *Theory of Linear Operations* currently available through Dover Publications. The relevant statement is the first theorem in Chapter 3. For a modern treatment, see section 2.11 of Walter Rudin's *Functional Analysis*; a specialization of the theorem proved there is that a continuous linear map between Fréchet spaces (locally convex topological vector spaces equipped with complete translation-invariant metrics) either is an open surjection or has image of first category.

Proof of Theorem 2 using probability. The idea of the second proof is to show that with probability 1, a randomly chosen power series that converges in D is noncontinuable.[8] As a warm-up, consider the case of the unit disc in $\mathbb{C}$. Suppose that the series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence equal to 1. The claim is that $\sum_{n=0}^{\infty} \pm c_n z^n$ has the unit circle as natural boundary for almost all choices of the plus-or-minus signs.

The statement can be made precise by introducing the Rademacher functions. When n is a nonnegative integer, the Rademacher function $\epsilon_n(t)$ is defined on the interval $[0, 1]$ as follows:

$$\epsilon_n(t) = \operatorname{sgn}\sin(2^n \pi t) = \begin{cases} 1, & \text{if } \sin(2^n\pi t) > 0, \\ -1, & \text{if } \sin(2^n\pi t) < 0, \\ 0, & \text{if } \sin(2^n\pi t) = 0. \end{cases}$$

Alternatively, the Rademacher functions can be described in terms of binary expansions. Suppose a number t between 0 and 1 is written in binary form as $\sum_{n=1}^{\infty} a_n(t)/2^n$. Then $\epsilon_n(t) = 1 - 2a_n(t)$, except for the finitely many rational values of t that can be written with denominator 2^n (which in any case are values of t for which $a_n(t)$ is not well defined).

Exercise 8. Show that the Rademacher functions form an orthonormal system in the space $L^2[0, 1]$ of square-integrable, real-valued functions. Are the Rademacher functions a *complete* orthonormal system?

The Rademacher functions provide a mathematical model for the notion of "random plus and minus signs." In the language of probability theory, the Rademacher functions are independent and identically distributed symmetric random variables. Each function takes the value $+1$ with probability $1/2$, the value -1 with probability $1/2$, and the value 0 on a set of measure zero (in fact, on a finite set). The intuitive meaning of "independence" is that knowing the value of one particular Rademacher function gives no information about the value of any other Rademacher function.

Here is a precise version of the statement about random series being noncontinuable.[9]

Theorem 3 (Paley–Zygmund)**.** *If the power series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence* 1*, then for almost every value of t in* $[0, 1]$*, the power series $\sum_{n=0}^{\infty} \epsilon_n(t) c_n z^n$ has the unit circle as natural boundary.*

The words "almost every" mean, as usual, that the exceptional set is a subset of $[0, 1]$ having measure zero. In probabilists' language, one says that the power series "almost surely" has the unit circle as natural boundary. Implicit in the conclusion is that the radius of convergence of the power series $\sum_{n=0}^{\infty} \epsilon_n(t) c_n z^n$ is almost surely equal to 1; this property is evident since the radius of convergence depends only on the moduli of the coefficients in the series, and almost surely $|\epsilon_n(t) c_n| = |c_n|$ for every n.

[8] A reference for this section is Jean-Pierre Kahane, *Some Random Series of Functions*, Cambridge University Press; see especially Chapter 4.

[9] R. E. A. C. Paley and A. Zygmund, On some series of functions, *Proceedings of the Cambridge Philosophical Society* **26** (1930), no. 3, 337–357 (announcement of the theorem without proof); **28** (1932), no. 2, 190–205 (proof of the theorem).

Proof. It will suffice to show for an arbitrary point p on the unit circle that the set of points t in the unit interval for which the power series $\sum_{n=0}^{\infty} \epsilon_n(t) c_n z^n$ continues analytically across p is a set of measure zero. Indeed, take a countable set of points $\{p_j\}_{j=1}^{\infty}$ that is dense in the unit circle: the union over j of the corresponding exceptional sets of measure zero is still a set of measure zero, and when t is in the complement of this set, the power series $\sum_{n=0}^{\infty} \epsilon_n(t) c_n z^n$ is nowhere continuable.

So fix a point p on the unit circle. A technicality needs to be checked: is the set of points t for which the power series $\sum_{n=0}^{\infty} \epsilon_n(t) c_n z^n$ continues analytically to a neighborhood of p a *measurable* subset of the interval [0, 1]? In probabilists' language, the question is whether continuability across p is an *event*. The answer is affirmative for the following reason.

A holomorphic function f on the unit disc extends analytically across the boundary point p if and only if there is some rational radius r greater than $1/2$ such that the Taylor series of f centered at the point $p/2$ has radius of convergence greater than r. An equivalent statement is that

$$\limsup_{k\to\infty} (|f^{(k)}(p/2)|/k!)^{1/k} < 1/r,$$

or that there exists a positive rational number s less than 2 and a natural number N such that

$$|f^{(k)}(p/2)| < k!\, s^k \qquad \text{whenever } k > N.$$

If $f_t(z)$ denotes the series $\sum_{n=0}^{\infty} \epsilon_n(t) c_n z^n$, then

$$|f_t^{(k)}(p/2)| = \left| \sum_{n=k}^{\infty} \epsilon_n(t) c_n \frac{n!}{(n-k)!} (p/2)^{n-k} \right|.$$

This (absolutely) convergent series is a measurable function of t since each $\epsilon_n(t)$ is a measurable function, so the set of t in the interval [0, 1] for which $|f_t^{(k)}(p/2)| < k!\, s^k$ is a measurable set, say E_k. The set of points t for which the power series $\sum_{n=0}^{\infty} \epsilon_n(t) c_n z^n$ extends across the point p is then

$$\bigcup_{\substack{0<s<2 \\ s\in\mathbb{Q}}} \bigcup_{N\geq 1} \bigcap_{k>N} E_k,$$

which again is a measurable set, being obtained from measurable sets by countably many operations of taking intersections and unions.

Notice too that extendability of $\sum_{n=0}^{\infty} \epsilon_n(t) c_n z^n$ across the boundary point p is a "tail event": the property is insensitive to changing any finite number of terms of the series. A standard result from probability known as *Kolmogorov's zero–one law* implies that this event either has probability 0 or has probability 1.

Moreover, each Rademacher function has the same distribution as its negative (both ϵ_n and $-\epsilon_n$ take the value 1 with probability $1/2$ and the value -1 with probability $1/2$), so a property that is almost sure for the series $\sum_{n=0}^{\infty} \epsilon_n(t) c_n z^n$ is almost sure for the series $\sum_{n=0}^{\infty} (-1)^n \epsilon_n(t) c_n z^n$ or for any similar series obtained by changing the signs according to a fixed pattern that is independent of t. The intuition is that if S is a measurable subset of [0, 1], and each element t of S

is represented as a binary expansion $\sum_{n=1}^{\infty} a_n(t)/2^n$, then the set S' obtained by systematically flipping the bit $a_5(t)$ from 0 to 1 or from 1 to 0 has the same measure as the original set S; and similarly if multiple bits are flipped simultaneously.

Now suppose, seeking a contradiction, that there is a neighborhood U of p to which the power series $\sum_{n=0}^{\infty} \epsilon_n(t)c_n z^n$ continues analytically with probability 1. This neighborhood contains, for some natural number k, an arc of the unit circle of length greater than $2\pi/k$. For each nonnegative integer n, set b_n equal to -1 if n is a multiple of k and $+1$ otherwise. By the preceding observation, the power series $\sum_{n=0}^{\infty} b_n\epsilon_n(t)c_n z^n$ extends to U with probability 1. The difference of two continuable series is continuable, so the power series $\sum_{j=0}^{\infty} \epsilon_{jk}(t)c_{jk}z^{jk}$ containing only powers of z divisible by k also continues to the neighborhood U with probability 1. This new series is invariant under rotation by every integral multiple of angle $2\pi/k$, so this series almost surely continues analytically to a neighborhood of the whole unit circle. In other words, the power series $\sum_{j=0}^{\infty} \epsilon_{jk}(t)c_{jk}z^{jk}$ almost surely has radius of convergence greater than 1. Fix a natural number ℓ between 1 and $k-1$ and repeat the argument, changing b_n to be equal to -1 if n is congruent to ℓ modulo k and 1 otherwise. It follows that the power series $\sum_{j=0}^{\infty} \epsilon_{jk+\ell}(t)c_{jk+\ell}z^{jk+\ell}$, which equals z^ℓ times the rotationally invariant series $\sum_{j=0}^{\infty} \epsilon_{jk+\ell}(t)c_{jk+\ell}z^{jk}$, almost surely has radius of convergence greater than 1. Adding these series for the different residue classes modulo k recovers the original random series $\sum_{n=0}^{\infty} \epsilon_n(t)c_n z^n$, which therefore has radius of convergence greater than 1 almost surely. But as observed just before the proof, the radius of convergence of $\sum_{n=0}^{\infty} \epsilon_n(t)c_n z^n$ is almost surely equal to 1. The contradiction shows that the power series $\sum_{n=0}^{\infty} \epsilon_n(t)c_n z^n$ does, after all, have the unit circle as natural boundary almost surely. □

Now consider the multidimensional situation: suppose that D is the domain of convergence in $\mathbb{C}^n$ of the power series $\sum_\alpha c_\alpha z^\alpha$. Let ϵ_α denote one of the Rademacher functions, a different one for each multi-index α. The goal is to show that almost surely, the power series $\sum_\alpha \epsilon_\alpha(t)c_\alpha z^\alpha$ continues analytically across no boundary point of D. It suffices to show for one fixed boundary point p with nonzero coordinates that the series almost surely is singular at p; one gets the full conclusion as before by considering a countable dense sequence in the boundary.

Having fixed such a boundary point p, observe that if δ is an arbitrary positive number, then the power series $\sum_\alpha c_\alpha z^\alpha$ fails to converge absolutely at the dilated point $(1+\delta)p$; for in the contrary case, the series would converge absolutely in the whole polydisc centered at 0 determined by the point $(1+\delta)p$, so p would be in the interior of the convergence domain D instead of on the boundary. (The assumption that all coordinates of p are nonzero is used here.) Consequently, there are infinitely many values of the multi-index α for which $|c_\alpha[(1+2\delta)p]^\alpha| > 1$; for otherwise, the series $\sum_\alpha c_\alpha[(1+\delta)p]^\alpha$ would converge absolutely by comparison with the convergent geometric series $\sum_\alpha[(1+\delta)/(1+2\delta)]^{|\alpha|}$. In other words, there are infinitely many values of α for which $|c_\alpha p^\alpha| > 1/(1+2\delta)^{|\alpha|}$.

Now consider the single-variable random power series obtained by restricting the multi-variable random power series to the complex line through p. This series, as a function of λ in the unit disc in $\mathbb{C}$, is $\sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} \epsilon_\alpha(t)c_\alpha p^\alpha\right)\lambda^k$. The goal is to show that this single-variable power series almost surely has radius of convergence equal to 1 and almost surely is singular at the point on the unit circle where $\lambda = 1$. It then follows that the multi-variable random series $\sum_\alpha \epsilon_\alpha(t) c_\alpha z^\alpha$ almost surely is singular at p.

The deduction that the one-variable series almost surely is singular at 1 follows from the same argument used in the proof of the Paley–Zygmund theorem. Although the series coefficient $\sum_{|\alpha|=k} \epsilon_\alpha(t) c_\alpha p^\alpha$ is no longer a Rademacher funtion, it is still a symmetric random variable (symmetric means that the variable is equally distributed with its negative), and the coefficients for different values of k are independent, so the same proof applies.

What remains to show, then, is that the single-variable power series almost surely has radius of convergence equal to 1. The new goal is to obtain information about the size of the coefficients $\sum_{|\alpha|=k} \epsilon_\alpha(t) c_\alpha p^\alpha$ from the knowledge that $|c_\alpha p^\alpha| > 1/(1+2\delta)^{|\alpha|}$ for infinitely many values of α.

The orthonormality of the Rademacher functions implies that

$$\int_0^1 \Bigl| \sum_{|\alpha|=k} \epsilon_\alpha(t) c_\alpha p^\alpha \Bigr|^2 \, dt = \sum_{|\alpha|=k} |c_\alpha p^\alpha|^2.$$

The sum on the right-hand side is at least as large as any single term, so there are infinitely many values of k for which

$$\int_0^1 \Bigl| \sum_{|\alpha|=k} \epsilon_\alpha(t) c_\alpha p^\alpha \Bigr|^2 \, dt > \frac{1}{(1+2\delta)^{2k}}.$$

The issue now is to deduce some control on the function $\bigl|\sum_{|\alpha|=k} \epsilon_\alpha(t) c_\alpha p^\alpha\bigr|^2$ from the lower bound on its integral.

Lemma 2. If $g\colon [0,1] \to \mathbb{R}$ is a non-negative, square-integrable function, then the Lebesgue measure of the set of points at which the value of g is greater than or equal to $\frac{1}{2}\int_0^1 g(t)\,dt$ is at least

$$\frac{\left(\int_0^1 g(t)\,dt\right)^2}{4\int_0^1 g(t)^2\,dt}. \tag{2.2}$$

Proof. Let S denote the indicated subset of $[0,1]$ and μ its measure. On the set $[0,1] \setminus S$, the function g is bounded above by the constant $\frac{1}{2}\int_0^1 g(t)\,dt$, so

$$\begin{aligned} \int_0^1 g(t)\,dt &= \int_S g(t)\,dt + \int_{[0,1]\setminus S} g(t)\,dt \\ &\le \int_S g(t)\,dt + (1-\mu)\tfrac{1}{2}\int_0^1 g(t)\,dt \\ &\le \int_S g(t)\,dt + \tfrac{1}{2}\int_0^1 g(t)\,dt. \end{aligned}$$

Therefore

$$\tfrac{1}{4}\left(\int_0^1 g(t)\,dt\right)^2 \le \left(\int_S g(t)\,dt\right)^2.$$

By the Cauchy–Schwarz inequality,

$$\left(\int_S g(t)\,dt\right)^2 \le \mu \int_S g(t)^2\,dt \le \mu \int_0^1 g(t)^2\,dt.$$

Combining the preceding two inequalities yields the desired conclusion (2.2). $\square$

Now apply the lemma with $g(t)$ equal to $\left|\sum_{|\alpha|=k} \epsilon_\alpha(t) c_\alpha p^\alpha\right|^2$. The integral in the denominator of (2.2) equals

$$\int_0^1 \left|\sum_{|\alpha|=k} \epsilon_\alpha(t) c_\alpha p^\alpha\right|^4 dt. \tag{2.3}$$

Exercise 9. The integral of the product of four Rademacher functions equals 0 unless either all four functions are the same or the four functions are equal in pairs.

There are three ways to group four items into two pairs, so the integral (2.3) equals

$$\sum_{|\alpha|=k} |c_\alpha p^\alpha|^4 + 3 \sum_{\substack{|\alpha|=k\\|\beta|=k\\\alpha\ne\beta}} |c_\alpha p^\alpha|^2\,|c_\beta p^\beta|^2.$$

This expression is at most $3\left(\sum_{|\alpha|=k} |c_\alpha p^\alpha|^2\right)^2$, or $3\left(\int_0^1 g(t)\,dt\right)^2$. Accordingly, the quotient in (2.2) is bounded below by $1/12$ for the indicated choice of g. (The specific value $1/12$ is not significant; what matters is that it is a positive constant.)

The upshot is that there are infinitely many values of k for which there exists a subset of the interval $[0, 1]$ of measure at least $1/12$ such that

$$\left|\sum_{|\alpha|=k} \epsilon_\alpha(t) c_\alpha p^\alpha\right|^{1/k} > \frac{1}{2^{1/\{2k\}}(1+2\delta)}$$

for every t in this subset. The right-hand side exceeds $1/(1+3\delta)$ when k is sufficiently large. For different values of k, the expressions on the left-hand side are independent functions; the probability that two independent events occur simultaneously is the product of their probabilities. Accordingly, if m is a natural number, and m of the indicated values of k are selected, the probability is at most $(11/12)^m$ that there is *none* for which

$$\left|\sum_{|\alpha|=k} \epsilon_\alpha(t) c_\alpha p^\alpha\right|^{1/k} > \frac{1}{(1+3\delta)}. \tag{2.4}$$

Since $(11/12)^m$ tends to 0 as m tends to infinity, the probability is 1 that inequality (2.4) holds for *some* value of k. For an arbitrary natural number N, the same conclusion holds (for the same reason) for some value of k larger than N. The intersection of countably many sets of probability 1 is again a set of probability 1, so

$$\limsup_{k\to\infty}\left|\sum_{|\alpha|=k}\epsilon_\alpha(t)c_\alpha p^\alpha\right|^{1/k} \geq \frac{1}{(1+3\delta)}$$

with probability 1. (The argument in this paragraph is nothing but the proof of the standard *Borel–Cantelli lemma* from probability theory.)

Thus the one-variable power series $\sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k}\epsilon_\alpha(t)c_\alpha p^\alpha\right)\lambda^k$ almost surely has radius of convergence bounded above by $1+3\delta$. But δ is an arbitrary positive number, so the radius of convergence is almost surely bounded above by 1. The radius of convergence is surely no smaller than 1, for the series converges absolutely when $|\lambda|<1$. Therefore the radius of convergence is almost surely equal to 1. This conclusion completes the proof. □

Open Problem. *Prove the Cartan–Thullen theorem by using a multi-variable gap series (avoiding both probabilistic methods and the Baire category theorem).*

2.5 Summary: domains of convergence

The preceding discussion shows that for complete Reinhardt domains, the following properties are all equivalent.

- The domain is logarithmically convex.
- The domain is the domain of convergence of some power series.
- The domain is a domain of holomorphy.

In other words, the problem of characterizing domains of holomorphy is solved for the special case of complete Reinhardt domains.

2.6 The Hartogs phenomenon

In the preceding sections, the infinite series have been Maclaurin series. Consideration of Laurent series leads to an interesting new instance of automatic analytic continuation.

Theorem 4. *Suppose δ is a positive number less than 1, and f is holomorphic on $\{(w,z) : |w|<\delta$ and $|z|<1\}\cup\{(w,z) : 1-\delta<|z|<1$ and $|w|<1\}$. Then f extends to be holomorphic on the unit bidisc in $\mathbb{C}^2$.*

Notice that the initial domain of definition of f is a multi-circular (Reinhardt) domain, but not a *complete* Reinhardt domain. There is no loss of generality in considering a symmetric "Hartogs figure," for an asymmetric figure can be shrunk to obtain a symmetric one. It will be evident from the proof that the theorem generalizes routinely to dimensions greater than 2.

Proof. On each slice where w is fixed, the function $f(w,z)$ is holomorphic at least in an annulus of inner radius $1-\delta$ and outer radius 1, so $f(w,z)$ can be expanded in a Laurent series $\sum_{k=-\infty}^{\infty} c_k(w)z^k$. Moreover, if γ is an arbitrary positive number less than δ, then

$$c_k(w) = \frac{1}{2\pi i}\int_{|z|=1-\gamma} \frac{f(w,z)}{z^{k+1}}\,dz.$$

Since $f(w,z)$ is holomorphic when $1-\delta < |z| < 1$ and $|w| < 1$, this integral representation shows that each $c_k(w)$ is a holomorphic function of w in the unit disc.

When $|w| < \delta$, the Laurent series for $f(w,z)$ is actually a Maclaurin series: when $k < 0$, the coefficient $c_k(w)$ is identically equal to zero when $|w| < \delta$. By the one-dimensional identity theorem, the holomorphic function $c_k(w)$ is identically equal to zero in the whole unit disc when $k < 0$. Therefore the series expansion for $f(w,z)$ reduces to $\sum_{k=0}^{\infty} c_k(w)z^k$, a Maclaurin series for every value of w in the unit disk. This series defines the required holomorphic extension of f, assuming that the series converges uniformly on compact subsets of $\{\,(w,z) : |w| < 1$ and $|z| < 1-\gamma\,\}$.

To verify the normal convergence, fix an arbitrary positive number β less than 1, and observe that the continuous function $|f(w,z)|$ has some finite upper bound M on the compact set where $|w| \le \beta$ and $|z| = 1-\gamma$. Estimating the integral representation for the series coefficient shows that $|c_k(w)| \le M/(1-\gamma)^k$ when $|w| \le \beta$. Consequently, if α is an arbitrary positive number less than $1-\gamma$, then the series $\sum_{k=0}^{\infty} c_k(w)z^k$ converges absolutely when $|w| \le \beta$ and $|z| \le \alpha$ by comparison with the convergent geometric series $\sum_{k=0}^{\infty} M[\alpha/(1-\gamma)]^k$. Since the required locally uniform convergence holds, the series $\sum_{k=0}^{\infty} c_k(w)z^k$ does define the desired holomorphic extension of f to the bidisc. □

The same method of proof yields a result about "internal" analytic continuation rather than "external" analytic continuation.

Theorem 5. *If r is a positive radius less than 1, and f is holomorphic in the spherical shell $\{\,(w,z) \in \mathbb{C}^2 : r^2 < |w|^2 + |z|^2 < 1\,\}$, then f extends to be a holomorphic function on the whole unit ball.*

The theorem is stated in dimension 2 for convenience of exposition, but a corresponding result holds both in higher dimension and in other geometric settings. A general theorem (to be proved later) states that if K is a compact subset of an open set Ω in $\mathbb{C}^n$ (where $n \ge 2$), and $\Omega \setminus K$ is connected, then every holomorphic function on $\Omega \setminus K$ extends to be a holomorphic function on Ω. Theorems of this type are known collectively as "the Hartogs phenomenon."

In particular, Theorem 5 demonstrates that a holomorphic function of two complex variables cannot have an isolated singularity, for the function continues analytically across a compact hole in its domain. The same reasoning applied to the reciprocal of the function shows that a holomorphic function of two (or more) complex variables cannot have an isolated zero.

Proof of Theorem 5. As in the preceding proof, expand $f(w,z)$ for a fixed w as a Laurent series $\sum_{k=-\infty}^{\infty} c_k(w)z^k$. To see that each coefficient $c_k(w)$ is a holomorphic function of w in the unit disc is not quite as simple as before, because there is no evident *global* integral representation for $c_k(w)$. But holomorphicity is a local property, and it is evident that for each fixed w_0 in the unit disc, there is a neigborhood U of w_0 and a radius s such that the Cartesian product $U \times \{ z \in \mathbb{C} : |z| = s \}$ is contained in a compact subset of the spherical shell. Consequently, each $c_k(w)$ admits a local integral representation and therefore defines a holomorphic function on the unit disc.

When $|w|$ is close to 1, the Laurent series for $f(w, z)$ is a Maclaurin series, so when $k < 0$, the function $c_k(w)$ is identically equal to 0 on an open subset of the unit disc and hence on the whole disc. Therefore the series representation for $f(w, z)$ is a Maclaurin series for every w. That the series converges locally uniformly follows as before from the local integral representation for $c_k(w)$. Therefore the series defines the required holomorphic extension of $f(w, z)$ to the whole unit ball. □

2.7 Separate holomorphicity implies joint holomorphicity

The working definition of a holomorphic function of two (or more) variables is a continuous function that is holomorphic in each variable separately. Hartogs proved that the hypothesis of continuity is superfluous.

Theorem 6. *If $f(w, z)$ is holomorphic in w for each fixed z and holomorphic in z for each fixed w, then $f(w, z)$ is holomorphic jointly in the two variables; that is, $f(w, z)$ can be represented locally as a convergent power series in two variables.*

The analogous theorem holds also for functions of n complex variables with minor adjustments to the proof. But there is no corresponding theorem for functions of real variables. Indeed, the function on $\mathbb{R}^2$ that equals 0 at the origin and $xy/(x^2 + y^2)$ when $(x, y) \neq (0, 0)$ is real-analytic in each variable separately but is not even jointly continuous. A large literature exists about deducing properties that hold in all variables jointly from properties that hold in each variable separately.[10]

The proof of Hartogs depends on some prior work of William Fogg Osgood. Here is the initial step.[11]

Theorem 7 (Osgood, 1899)**.** *If $f(w, z)$ is bounded in both variables jointly and holomorphic in each variable separately, then $f(w, z)$ is holomorphic in both variables jointly.*

Proof. The conclusion is local and is invariant under translations and dilations of the coordinates, so there is no loss of generality in supposing that the domain of definition of f is the unit bidisc. Let M be an upper bound for the modulus of f in the bidisc.

[10] See a survey article by Marek Jarnicki and Peter Pflug, Directional regularity vs. joint regularity, *Notices of the American Mathematical Society* **58** (2011), no. 7, 896–904.

[11] W. F. Osgood, Note über analytische Functionen mehrerer Veränderlichen, *Mathematische Annalen* **52** (1899) 462–464.

For each fixed w, the function $f(w,z)$ can be expanded in a power series $\sum_{k=0}^{\infty} c_k(w)z^k$ that converges for z in the unit disc. Moreover, $|c_k(w)| \le M$ for each k by Cauchy's estimate. Accordingly, the series $\sum_{k=0}^{\infty} c_k(w)z^k$ actually converges uniformly in both variables jointly in an arbitrary compact subset of the bidisc. If each coefficient function $c_k(w)$ can be shown to be a holomorphic function of w, then it will follow that $f(w,z)$ is the locally uniform limit of jointly holomorphic functions, hence is jointly holomorphic.

Now $c_0(w) = f(w,0)$, so $c_0(w)$ is a holomorphic function of w in the unit disc by the hypothesis of separate holomorphicity. Proceed by induction. Suppose, for some natural number k, that $c_j(w)$ is a holomorphic function of w when $j < k$. Observe that

$$c_k(w) + \sum_{m=1}^{\infty} c_{k+m}(w)z^m = \frac{f(w,z) - \sum_{j=0}^{k-1} c_j(w)z^j}{z^k} \qquad \text{when } z \neq 0.$$

When z tends to 0, the left-hand side converges (uniformly with respect to w) to $c_k(w)$, hence so does the right-hand side. For every fixed nonzero value of z, the right-hand side is a holomorphic function of w by the induction hypothesis and the hypothesis of separate holomorphicity. Accordingly, the function $c_k(w)$ is the uniform limit of holomorphic functions, hence is holomorphic. That conclusion completes the induction argument and also the proof of the theorem. $\square$

Subsequently, Osgood made further progress but did achieve the ultimate result.[12]

Theorem 8 (Osgood, 1900)**.** *If $f(w,z)$ is holomorphic in each variable separately, then there is a dense open subset of the domain of f on which f is holomorphic in both variables jointly.*

Proof. It suffices to show that if $D_1 \times D_2$ is an arbitrary closed bidisc contained in the domain of definition of f, then there is an open subset of $D_1 \times D_2$ on which f is holomorphic. Let E_k denote the set of w in D_1 such that $|f(w,z)| \le k$ when $z \in D_2$. The continuity of $|f(w,z)|$ in w for z fixed implies that E_k is a closed subset of D_1: namely, for each fixed z, the set $\{\, w \in D_1 : |f(w,z)| \le k \,\}$ is closed, and E_k is the intersection of these closed sets as z runs over D_2. Moreover, every point of D_1 is contained in some E_k. By the Baire category theorem, there is some value of k for which the set E_k has interior points. Consequently, there is an open subset of $D_1 \times D_2$ on which f is bounded, hence holomorphic by Osgood's previous theorem. $\square$

Proof of Theorem 6 on separate holomorphicity. In view of Theorem 8 and the local nature of the conclusion, it suffices to prove that if $f(w,z)$ is separately holomorphic on a neighborhood of the closed unit bidisc, and there exists a positive δ less than 1 such that f is jointly holomorphic in a neighborhood of the smaller bidisc where $|z| \le \delta$ and $|w| \le 1$, then f is jointly holomorphic on the open unit bidisc.

[12] W. F. Osgood, Zweite Note über analytische Functionen mehrerer Veränderlichen, *Mathematische Annalen* **53** (1900) 461–464.

In this situation, write $f(w,z)$ as a series $\sum_{k=0}^{\infty} c_k(w)z^k$. Each coefficient function $c_k(w)$ can be written as an integral

$$\frac{1}{2\pi i}\int_{|z|=\delta}\frac{f(w,z)}{z^{k+1}}\,dz$$

and so is a holomorphic function of w in the unit disc. If M is an upper bound for $|f(w,z)|$ when $|z|\le\delta$ and $|w|\le 1$, then $|c_k(w)|\le M/\delta^k$ for every k. Accordingly, there is a (large) constant B such that $|c_k(w)|^{1/k} < B$ for every k. Moreover, for each fixed w, the series $\sum_{k=0}^{\infty} c_k(w)z^k$ converges for z in the unit disc, so $\limsup_{k\to\infty}|c_k(w)|^{1/k}\le 1$.

The goal now is to show that if ε is an arbitrary positive number, and r is an arbitrary radius slightly less than 1, then there exists a natural number N such that $|c_k(w)|^{1/k} < 1+\varepsilon$ when $k\ge N$ and $|w|\le r$. This property implies that the series $\sum_{k=0}^{\infty} c_k(w)z^k$ converges uniformly on compact subsets of the open unit bidisc, whence $f(w,z)$ is jointly holomorphic on the bidisc. Letting $u_k(w)$ denote the subharmonic function $|c_k(w)|^{1/k}$ reduces the problem to the following technical lemma, after which the proof will be complete. □

Lemma 3. Suppose $\{u_k\}_{k=1}^{\infty}$ is a sequence of subharmonic functions on the unit disc that are uniformly bounded above by a (large) constant B, and suppose $\limsup_{k\to\infty}u_k(w)\le 1$ for every w in the unit disc. Then for every positive ε and every radius r less than 1, there exists a natural number N such that $u_k(w)\le 1+\varepsilon$ when $|w|\le r$ and $k\ge N$.

Proof. A compactness argument reduces the problem to showing that for each point w_0 in the disk of radius r, there is a neighborhood U of w_0 and a natural number N such that $u_k(w)\le 1+\varepsilon$ when $w\in U$ and $k\ge N$. The definition of lim sup provides a natural number N such that $u_k(w_0) < 1+\frac{1}{2}\varepsilon$ when $k\ge N$. The issue is to get an analogous inequality that is locally uniform in w.

Since u_k is upper semicontinuous, there is a neighborhood of w_0 in which u_k remains less than $1+\varepsilon$, but the size of this neighborhood might depend on k. The key idea for getting an estimate independent of k is to apply the subaveraging property of subharmonic functions, for the uniform upper bound on the functions means that integrals over discs are stable under perturbations of the center point. Here are the details.

Fix a positive number δ less than $(1-r)/2$. Fatou's lemma implies that

$$\int_{|w-w_0|<\delta}\liminf_{k\to\infty}(B-u_k(w))\,d\mathrm{Area}_w \le \liminf_{k\to\infty}\int_{|w-w_0|<\delta}(B-u_k(w))\,d\mathrm{Area}_w,$$

so canceling $B\pi\delta^2$, changing the signs, and invoking the hypothesis shows that

$$\pi\delta^2 \ge \int_{|w-w_0|<\delta}\limsup_{k\to\infty}u_k(w)\,d\mathrm{Area}_w \ge \limsup_{k\to\infty}\int_{|w-w_0|<\delta}u_k(w)\,d\mathrm{Area}_w.$$

Accordingly, there is a natural number N such that

$$\int_{|w-w_0|<\delta}u_k(w)\,d\mathrm{Area}_w < \left(1+\tfrac{1}{2}\varepsilon\right)\pi\delta^2 \qquad \text{when } k\ge N.$$

If γ is a positive number less than δ, and w_1 is a point such that $|w_1 - w_0| < \gamma$, then the disc of radius $\delta + \gamma$ centered at w_1 contains the disc of radius δ centered at w_0, with an excess of area equal to $\pi(\gamma^2 + 2\gamma\delta)$. The subaveraging property of subharmonic functions implies that

$$\pi(\delta+\gamma)^2 u_k(w_1) \le \int_{|w-w_1|<\delta+\gamma} u_k(w)\, d\text{Area}_w < \left(1+\tfrac{1}{2}\varepsilon\right)\pi\delta^2 + B\pi\left(\gamma^2+2\gamma\delta\right)$$

when $k \ge N$, or

$$u_k(w_1) < \frac{\left(1+\frac{1}{2}\varepsilon\right)\pi\delta^2 + B\pi(\gamma^2+2\gamma\delta)}{\pi(\delta+\gamma)^2}.$$

The right-hand side has limit as $\gamma \to 0$ equal to $1 + \frac{1}{2}\varepsilon$, so there is a small positive value of γ for which $u_k(w_1) < 1 + \varepsilon$ when $k \ge N$ and w_1 is an arbitrary point in the disk of radius γ centered at w_0. This locally uniform estimate completes the proof of the lemma. □

Exercise 10. Prove that a separately polynomial function on $\mathbb{C}^2$ is a jointly polynomial function.[13]

Exercise 11. What adjustments are needed in the proof to obtain the analogue of Theorem 6 in dimension n?

Exercise 12. Define $f : \mathbb{C}^2 \to \mathbb{C} \cup \{\infty\}$ as follows:

$$f(w,z) = \begin{cases} (w+z)^2/(w-z), & \text{when } w \ne z; \\ \infty, & \text{when } w = z \text{ but } (w,z) \ne (0,0); \\ 0, & \text{when } (w,z) = (0,0). \end{cases}$$

Show that f is separately meromorphic, and $f(0,0)$ is finite, yet f is not continuous at $(0,0)$ (not even with respect to the spherical metric on the extended complex numbers).[14]

[13]The corresponding statement for functions on $\mathbb{R}^2$ was proved by F. W. Carroll, A polynomial in each variable separately is a polynomial, *American Mathematical Monthly* **68** (1961) 42.

[14]This example is due to Theodore J. Barth, Families of holomorphic maps into Riemann surfaces, *Transactions of the American Mathematical Society* **207** (1975) 175–187. Barth's interpretation of the example is that f is a holomorphic *mapping* from $\mathbb{C}^2$ into the Riemann sphere (a one-dimensional, compact, complex manifold) that is separately holomorphic but not jointly holomorphic.

3 Convexity

From one point of view, convexity is an unnatural property in complex analysis. The Riemann mapping theorem shows that already in dimension 1, convexity is not preserved by biholomorphic mappings: indeed, every nonconvex but simply connected domain in the plane is conformally equivalent to the unit disc.

On the other hand, section 2.2 reveals that a special kind of convexity—namely, logarithmic convexity—appears naturally in studying convergence domains of power series. Various analogues of convexity turn out to be central to some fundamental problems in multi-dimensional complex analysis.

3.1 Real convexity

Ordinary geometric convexity can be described either through an internal geometric property—the line segment joining two points of the set stays within the set—or through an external property—every point outside the set can be separated from the set by a hyperplane. The latter geometric property can be rephrased in analytic terms by saying that every point outside the set can be separated from the set by a linear function; that is, there is a linear function that is larger at the specified exterior point than anywhere on the set.

For an arbitrary set, not necessarily convex, its *convex hull* is the smallest convex set containing it, that is, the intersection of all convex sets containing it. The convex hull of an open set is open, and in $\mathbb{R}^n$ (or in any finite-dimensional vector space), the convex hull of a compact set is compact.[1]

Observe that an open set G in $\mathbb{R}^n$ is convex if and only if the convex hull of every compact subset K is again a compact subset of G. Indeed, if K is a subset of G, then the convex hull of K is a subset of the convex hull of G, so if G is already convex, then the convex hull of K is both compact and a subset of G. Conversely, if G is not convex, then there are two points of G such that the line segment joining them goes outside of G; take K to be the union of the two points.

[1]In an infinite-dimensional Hilbert space, the convex hull of a compact set is not necessarily closed, let alone compact. But the *closure* of the convex hull of a compact set is compact in every Hilbert space and in every Banach space. See, for example, Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third edition, Springer, 2006, section 5.6.

A standard reference for the finite-dimensional theory of real convexity is R. Tyrrell Rockafellar, *Convex Analysis*, Princeton University Press, 1970 (reprinted 1997).

3.2 Convexity with respect to a class of functions

The analytic description of convexity has a natural generalization. Suppose that $\mathcal{F}$ is a class of upper semi-continuous real-valued functions on an open set G in $\mathbb{C}^n$ (which might be $\mathbb{C}^n$ itself). [Recall that a real-valued function f is upper semi-continuous if $f^{-1}(-\infty, a)$ is an open set for every real number a. Upper semi-continuity guarantees that f attains a maximum on each compact set.] A compact subset K of G is called convex with respect to the class $\mathcal{F}$ if for every point p of $G \setminus K$ there exists an element f of $\mathcal{F}$ for which $f(p) > \max_{z \in K} f(z)$; in other words, every point outside K can be separated from K by a function in $\mathcal{F}$. If $\mathcal{F}$ is a class of functions that are complex-valued but not real-valued (holomorphic functions, say), then it is natural to consider convexity with respect to the class of absolute values of the functions in $\mathcal{F}$; one typically says simply "$\mathcal{F}$-convex" for short when the meaning is really "$\mathcal{G}$-convex, where $\mathcal{G} = \{\, |f| : f \in \mathcal{F}\,\}$."

The $\mathcal{F}$-convex hull of a compact set K, denoted by $\widehat{K}_{\mathcal{F}}$ (or simply $\widehat{K}$ if the class $\mathcal{F}$ is understood), is the smallest $\mathcal{F}$-convex set containing K (assuming, of course, that there *are* some $\mathcal{F}$-convex sets containing K). One says that an open subset Ω of G is $\mathcal{F}$-convex if for every compact subset K of Ω, the $\mathcal{F}$-convex hull $\widehat{K}_{\mathcal{F}}$ is again a compact subset of Ω.

Example 1. Let G be $\mathbb{R}^n$, and let $\mathcal{F}$ be the set of all continuous functions on $\mathbb{R}^n$. Then every compact set K is $\mathcal{F}$-convex because, by Urysohn's lemma, every point not in K can be separated from K by a continuous function.

Example 2. Let G be $\mathbb{C}^n$, and let $\mathcal{F}$ be the set of coordinate functions, $\{z_1, \dots, z_n\}$. The $\mathcal{F}$-convex hull of a single point w is the set of all points z for which $|z_j| \le |w_j|$ for all j, that is, the polydisc determined by the point w. (If some coordinate of w is equal to 0, then the polydisc is degenerate.) More generally, the $\mathcal{F}$-convex hull of a compact set K is the set of points z for which $|z_j| \le \max\{\, |\zeta_j| : \zeta \in K\,\}$ for every j. Consequently, the $\mathcal{F}$-convex open sets are precisely the open polydiscs centered at the origin.

Exercise 13. Show that a domain is convex with respect to the class $\mathcal{F}$ consisting of the monomials z^α if and only if the domain is a logarithmically convex, complete Reinhardt domain.

A useful observation is that increasing the class of functions $\mathcal{F}$ makes it easier to separate points, so the collection of $\mathcal{F}$-convex sets becomes larger. In other words, if $\mathcal{F}_1 \subset \mathcal{F}_2$, then every $\mathcal{F}_1$-convex set is also $\mathcal{F}_2$-convex.

Exercise 14. As indicated above, ordinary geometric convexity in $\mathbb{R}^n$ is the same as convexity with respect to the class of linear functions $a_1x_1 + \dots + a_nx_n$; moreover, it is equivalent to consider convexity with respect to the class of affine linear functions $a_0 + a_1x_1 + \dots + a_nx_n$. The aim of this exercise is to determine what happens if the functions are replaced with their absolute values.

1. Suppose $\mathcal{F}$ is the set $\{\, |a_1x_1 + \dots + a_nx_n| \,\}$ of absolute values of linear functions on $\mathbb{R}^n$. Describe the $\mathcal{F}$-convex hull of a general compact set.

2. Suppose $\mathcal{F}$ is the set $\{\, |a_0 + a_1x_1 + \dots + a_nx_n| \,\}$ of absolute values of affine linear functions. Describe the $\mathcal{F}$-convex hull of a general compact set.

Exercise 15. Repeat the preceding exercise in the setting of $\mathbb{C}^n$ and functions with complex coefficients:

1. Suppose $\mathcal{F}$ is the set $\{\, |a_1 z_1 + \cdots + a_n z_n| \,\}$ of absolute values of complex linear functions. Describe the $\mathcal{F}$-convex hull of a compact set.

2. Suppose $\mathcal{F}$ is the set $\{\, |a_0 + a_1 z_1 + \cdots + a_n z_n| \,\}$ of absolute values of affine complex linear functions. Describe the $\mathcal{F}$-convex hull of a compact set.

Observe that a point and a compact set can be separated by $|f|$ if and only they can be separated by $|f|^2$ or more generally by $|f|^k$. Hence there is no loss of generality in assuming that a class $\mathcal{F}$ of holomorphic functions is closed under forming positive integral powers. The next example demonstrates that the situation changes if arbitrary products are allowed.

3.2.1 Polynomial convexity

Again let G be all of $\mathbb{C}^n$, and let $\mathcal{F}$ be the set of polynomials (in the complex variables). Then $\mathcal{F}$-convexity is called *polynomial convexity*. (When the setting is $\mathbb{C}^n$, the word "polynomial" is usually understood to mean "holomorphic polynomial," that is, a polynomial in the complex coordinates $z_1, \ldots, z_n$ rather than a polynomial in the underlying real coordinates of $\mathbb{R}^{2n}$.)

A first observation is that the polynomial hull of a compact set is a subset of the ordinary convex hull. Indeed, if a point is separated from a compact set by a real-linear function $\operatorname{Re} \ell(z)$, then it is separated by $e^{\operatorname{Re} \ell(z)}$ and hence by $|e^{\ell(z)}|$; the entire function $e^{\ell(z)}$ can be approximated uniformly on compact sets by polynomials. (Alternatively, apply the solution of Exercise 15.)

When $n = 1$, polynomial convexity is a topological property. By Runge's approximation theorem, if K is a compact subset of $\mathbb{C}$ (not necessarily connected), and if K has no holes (that is, $\mathbb{C} \setminus K$ is connected), then every function that is holomorphic in a neighborhood of K can be approximated uniformly on K by (holomorphic) polynomials.[2] Now if K has no holes, and p is a point outside K, then Runge's theorem implies that the function equal to 0 in neighborhood of K and equal to 1 in a neighborhood of p can be arbitrarily well approximated on $K \cup \{p\}$ by polynomials; hence p is not in the polynomial hull of K. On the other hand, if K has a hole, then the maximum principle implies that points inside the hole are in the polynomial hull of K. In other words, a compact set K in $\mathbb{C}$ is polynomially convex if and only if K has no holes. A connected open subset of $\mathbb{C}$ is polynomially convex if and only if it is simply connected, that is, its complement with respect to the extended complex numbers is connected.

The story is much more complicated when $n > 1$, for polynomial convexity is no longer determined by a topological condition. For instance, whether or not a circle (of real dimension 1) is polynomially convex depends on how that curve is situated with respect to the complex structure of $\mathbb{C}^n$.

[2]There is a deeper theorem due to S. N. Mergelyan: namely, the conclusion follows from the weaker hypothesis that the function to be approximated is continuous on K and holomorphic at the interior points of K.

Example 3. (a) In $\mathbb{C}^2$, the circle $\{ (\cos\theta + i\sin\theta, 0) : 0 \le \theta \le 2\pi \}$ is not polynomially convex. The one-dimensional maximum principle implies that the polynomial hull of this curve is the disc $\{ (z_1, 0) : |z_1| \le 1 \}$.

(b) In $\mathbb{C}^2$, the circle $\{ (\cos\theta, \sin\theta) : 0 \le \theta \le 2\pi \}$ is polynomially convex. Indeed, since the polynomial hull is a subset of the ordinary convex hull, all that needs to be shown is that points inside the disc bounded by the circle can be separated from the circle by (holomorphic) polynomials. The polynomial $1 - z_1^2 - z_2^2$ is identically equal to 0 on the circle and takes positive real values at points inside the circle, so this polynomial exhibits the required separation.

The preceding idea can easily be generalized to produce a wider class of examples of polynomially convex sets.

Example 4. If K is a compact subset of the real subspace of $\mathbb{C}^n$ (that is, $K \subset \mathbb{R}^n \subset \mathbb{C}^n$), then K is polynomially convex.

This proposition can be proved by invoking the Weierstrass approximation theorem in $\mathbb{R}^n$, but it is interesting that there is an elementary, "bare-hands" argument. First notice that convexity with respect to (holomorphic) polynomials is the same property as convexity with respect to entire functions, since an entire function can be approximated uniformly on a compact set by polynomials (for instance, by the partial sums of the Maclaurin series). Therefore it suffices to write down an entire function whose modulus separates K from a specified point q outside of K.

A function that does the trick is the Gaussian function $\exp\sum_{j=1}^{n} -(z_j - \operatorname{Re} q_j)^2$. To see why, let $M(z)$ denote the modulus of this function: namely, $\exp\sum_{j=1}^{n} \left[(\operatorname{Im} z_j)^2 - (\operatorname{Re} z_j - \operatorname{Re} q_j)^2\right]$. If $q \notin \mathbb{R}^n$, then $M(q) = \exp\sum_{j=1}^{n} (\operatorname{Im} q_j)^2 > 1$, while

$$\max_{z\in K} M(z) = \max_{z\in K} \exp\sum_{j=1}^{n} -(z_j - \operatorname{Re} q_j)^2 \le 1.$$

On the other hand, if $q \in \mathbb{R}^n$ but $q \notin K$, then the expression $\sum_{j=1}^{n} (z_j - \operatorname{Re} q_j)^2$ has a positive lower bound on the compact set K, so $\max_{z\in K} M(z) < 1$, while $M(q) = 1$. The required separation holds in both cases. (Actually, it suffices to check the second case, for the polynomial hull of K is a subset of the convex hull of K and hence a subset of $\mathbb{R}^n$.)

Exercise 16. Show that every compact subset of a *totally real* subspace of $\mathbb{C}^n$ is polynomially convex. (A real subspace of $\mathbb{C}^n$ is called totally real if it contains no complex line. In other words, a subspace is totally real if, whenever z is a nonzero point in the subspace, the point iz is not in the subspace.)

Having some polynomially convex sets in hand, one can generate additional polynomially convex sets by applying the following example.

Example 5. If K is a polynomially convex compact subset of $\mathbb{C}^n$, and p is a polynomial, then the graph $\{ (z, p(z)) \in \mathbb{C}^{n+1} : z \in K \}$ is a polynomially convex compact subset of $\mathbb{C}^{n+1}$.

For suppose $\alpha \in \mathbb{C}^n$ and $\beta \in \mathbb{C}$, and (α, β) is not in the graph of p over K; to separate the point (α, β) from the graph by a polynomial, consider two cases. If $\alpha \notin K$, then there is a polynomial of n variables that separates α from K in $\mathbb{C}^n$; the same polynomial, viewed as a polynomial on $\mathbb{C}^{n+1}$ that is independent of z_{n+1}, separates the point (α, β) from the graph of p. Suppose, on the other hand, that $\alpha \in K$, but $\beta \neq p(\alpha)$. Then the polynomial $z_{n+1} - p(z)$ is identically equal to 0 on the graph and is not equal to 0 at (α, β), so this polynomial separates (α, β) from the graph.

Exercise 17. If f is a function that is continuous on the closed unit disc in $\mathbb{C}$ and holomorphic on the interior of the disc, then the graph of f in $\mathbb{C}^2$ is polynomially convex.

More generally, a smooth analytic disc—the image in $\mathbb{C}^n$ of the closed unit disc under a holomorphic embedding whose derivative is never equal to zero—is always polynomially convex.[3] But biholomorphic images of polydiscs can fail to be polynomially convex.[4]

The basic examples of polynomially convex sets in $\mathbb{C}^n$ with nonvoid interior are the so-called polynomial polyhedra: sets of the form $\{ z \in \mathbb{C}^n : |p_1(z)| \leq 1, \dots, |p_k(z)| \leq 1 \}$ or $\{ z \in \mathbb{C}^n : |p_1(z)| < 1, \dots, |p_k(z)| < 1 \}$, where each function p_j is a polynomial. The model case is the polydisc $\{ z \in \mathbb{C}^n : |z_1| \leq 1, \dots, |z_n| \leq 1 \}$; another concrete example is the logarithmically convex, complete Reinhardt domain $\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1, \text{ and } |2z_1z_2| < 1 \}$.

A polynomial polyhedron evidently is polynomially convex, since a point in the complement is separated from the polyhedron by at least one of the defining polynomials. Notice that k, the number of polynomials, can be larger than the dimension n. (If the polyhedron is compact and nonvoid, then the number k cannot be less than n, but proving this property requires some tools not yet introduced.[5]) A standard way to force a polynomial polyhedron to be bounded is to include in the set of defining polynomials the functions z_j/R for some large R and for each j from 1 to n.

A theorem from one-dimensional complex analysis known as *Hilbert's lemniscate theorem*[6] says that the boundary of a bounded, simply connected domain in $\mathbb{C}$ can be approximated within an arbitrary positive ε by a polynomial lemniscate: namely, by the set where some polynomial has constant modulus. An equivalent statement is that if K is a compact, polynomially convex subset of $\mathbb{C}$, and U is an open neighborhood of K, then there is a polynomial p such that $|p(z)| < 1$ when $z \in K$ and $|p(z)| > 1$ when $z \in \mathbb{C} \setminus U$.

The following statement is a generalization of Hilbert's lemniscate theorem to higher dimension: every polynomially convex set in $\mathbb{C}^n$ can be approximated by polynomial polyhedra.

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[3]John Wermer, An example concerning polynomial convexity, *Mathematische Annalen* **139** (1959) 147–150.

[4]For an example in $\mathbb{C}^3$, see John Wermer, Addendum to "An example concerning polynomial convexity," *Mathematische Annalen* **140** (1960) 322–323. For an example in $\mathbb{C}^2$, see John Wermer, On a domain equivalent to the bidisc, *Mathematische Annalen* **248** (1980), no. 3, 193–194.

[5]If w is a point of the polyhedron, then the k sets $\{ z \in \mathbb{C}^n : p_j(z) - p_j(w) = 0 \}$ are analytic varieties of codimension 1 that intersect in an analytic variety of dimension at least $n-k$ that is contained in the polyhedron. If $k < n$, then this analytic variety has positive dimension, but there are no compact analytic varieties of positive dimension.

[6]D. Hilbert, Ueber die Entwickelung einer beliebigen analytischen Function einer Variabeln in eine unendliche nach ganzen rationalen Functionen fortschreitende Reihe, *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* (1897) 63–70; `http://resolver.sub.uni-goettingen.de/purl?GDZPPN002497727`.

Theorem 9. *(a) If K is a compact polynomially convex set, and U is an open neighborhood of K, then there is an open polynomial polyhedron P such that $K \subset P \subset U$.*

(b) If G is a polynomially convex open set, then G can be written as the union of an increasing sequence of open polynomial polyhedra.

Proof. (a) Since the set K is bounded, it is contained in the interior of some closed polydisc D. For each point w in $D \setminus U$, there is a polynomial p that separates w from K. This polynomial can be multiplied by a suitable constant to guarantee that $\max\{\,|p(z)| : z \in K\,\} < 1 < |p(w)|$. Hence the set $\{\,z : |p(z)| < 1\,\}$ contains K and is disjoint from a neighborhood of w. Since the set $D \setminus U$ is compact, there are finitely many polynomials $p_1, \ldots, p_k$ such that the polyhedron $\bigcap_{j=1}^{k}\{\,z : |p_j(z)| < 1\,\}$ contains K and does not intersect $D \setminus U$. Cutting down this polyhedron by intersecting with D gives a new polyhedron that contains K and is contained in U.

(b) Exhaust G by an increasing sequence of compact sets. The polynomial hulls of these sets form another increasing sequence of compact subsets of G (since G is polynomially convex). After possibly omitting some of the sets and renumbering, one obtains an exhaustion of G by a sequence $\{K_j\}_{j=1}^{\infty}$ of polynomially convex compact sets such that each K_j is contained in the interior of K_{j+1}. The first part of the theorem then provides a sequence $\{P_j\}$ of open polynomial polyhedra such that $K_j \subset P_j \subset K_{j+1}$.

□

In Hilbert's lemniscate theorem, a single polynomial suffices. An interesting question is whether n polynomials suffice to define an approximating polyhedron in $\mathbb{C}^n$. A partial result in this direction has been known for half a century. A polyhedron can be a disconnected set; Errett Bishop showed[7] that the approximation can be accomplished by a set that is the union of a finite number of the connected components of a polyhedron defined by n polynomials. Even in the simple case of a ball, however, it is unknown whether a full polyhedron defined by n polynomials will serve as the approximating set.

Open Problem. *Can the closed unit ball in $\mathbb{C}^n$ be arbitrarily well approximated from outside by polynomial polyhedra defined by n polynomials?*

When $n = 2$, the answer to the preceding question is affirmative.[8] But when $n \geq 3$, the question remains open.[9]

[7] Errett Bishop, Mappings of partially analytic spaces, *American Journal of Mathematics* **83** (1961), no. 2, 209–242; `http://www.jstor.org/stable/2372953`.

[8] Thomas Bloom, Norman Levenberg, and Yu. Lyubarskii, A Hilbert lemniscate theorem in $\mathbb{C}^2$, *Annales de l'Institut Fourier* **58** (2008), no. 6, 2191-2220; `doi:10.5802/aif.2411`.

[9] Stéphanie Nivoche, Polynomial convexity, special polynomial polyhedra and the pluricomplex Green function for a compact set in $\mathbb{C}^n$, *Journal de Mathématiques Pures et Appliquées* **91** (2009), no. 4, 364–383; `doi:10.1016/j.matpur.2009.01.003`.

Although the theory of polynomial convexity is sufficiently mature that there exists a good reference book,[10] determining the polynomial hull of even quite simple sets remains a fiendishly difficult problem. In particular, the union of two disjoint, compact, polynomially convex sets need not be polynomially convex.

Example 6 (Kallin,[11] 1965). Let K_1 be $\{\, (e^{i\theta}, e^{-i\theta}) \in \mathbb{C}^2 : 0 \le \theta < 2\pi \,\}$, and let K_2 be $\{\, (2e^{i\theta}, \frac{1}{2}e^{-i\theta}) \in \mathbb{C}^2 : 0 \le \theta < 2\pi \,\}$. Both sets K_1 and K_2 are polynomially convex in view of Exercise 16, since K_1 lies in the totally real subspace of $\mathbb{C}^2$ in which $z_1 = \overline{z_2}$, and K_2 lies in the totally real subspace in which $z_1/4 = \overline{z_2}$. The union $K_1 \cup K_2$ is not polynomially convex, for the polynomial hull contains the set $\{\, (z, 1/z) : 1 < |z| < 2 \,\}$. Indeed, if $p(z_1, z_2)$ is a polynomial on $\mathbb{C}^2$ whose modulus is less than 1 on $K_1 \cup K_2$, then $p(z, 1/z)$ is a holomorphic function on $\mathbb{C} \setminus \{0\}$ whose modulus is less than 1 on the boundary of the annulus $\{\, z \in \mathbb{C} : 1 < |z| < 2 \,\}$ and hence (by the maximum principle) on the interior of the annulus.

Here is one accessible positive result: *If K_1 and K_2 are disjoint, compact, convex sets in $\mathbb{C}^n$, then the union $K_1 \cup K_2$ is polynomially convex.*

Proof. The disjoint convex sets K_1 and K_2 can be separated by a real hyperplane, or equivalently by the real part of a complex linear function ℓ. The geometric picture is that ℓ projects $\mathbb{C}^n$ onto a complex line (a one-dimensional complex subspace). The sets $\ell(K_1)$ and $\ell(K_2)$ are disjoint, compact, convex sets in $\mathbb{C}$.

Suppose now that w is a point outside of $K_1 \cup K_2$. The goal is to separate w from $K_1 \cup K_2$ by a polynomial.

If $\ell(w) \notin \ell(K_1) \cup \ell(K_2)$, then Runge's theorem provides a polynomial p of one complex variable such that $|p(\ell(w))| > 1$, while $|p(z)| < 1$ when $z \in \ell(K_1) \cup \ell(K_2)$. In other words, the polynomial $p \circ \ell$ separates w from $K_1 \cup K_2$ in $\mathbb{C}^n$.

If, on the other hand, $\ell(w) \in \ell(K_1) \cup \ell(K_2)$, then one may as well assume that $\ell(w) \in \ell(K_1)$. But $w \notin K_1$, and K_1 is polynomially convex, so there is a polynomial p on $\mathbb{C}^n$ such that $|p(w)| > 1$ and $|p(z)| < 1/3$ when $z \in K_1$. Let M be an upper bound for $|p|$ on K_2. Applying Runge's theorem in $\mathbb{C}$ gives a polynomial q of one variable such that $|q| < 1/(3M)$ on $\ell(K_2)$ and $2/3 \le |q| \le 1$ on $\ell(K_1)$. The product polynomial $p \times (q \circ \ell)$ separates w from $K_1 \cup K_2$: for on K_1, the first factor has modulus less than $1/3$, and the second factor has modulus no greater than 1; on K_2, the first factor has modulus at most M, and the second factor has modulus less than $1/(3M)$; and at w, the modulus of the first factor exceeds 1, and the modulus of the second factor is at least $2/3$. $\square$

The preceding proposition is a special case of a separation lemma of Eva Kallin, who showed in the cited paper that the union of three closed, pairwise disjoint balls in $\mathbb{C}^n$ is always polynomially convex. The question of the polynomial convexity of the union of four pairwise disjoint closed balls is still open after more than four decades. The problem is subtle, for Kallin

[10] Edgar Lee Stout, *Polynomial Convexity*, Birkhäuser Boston, 2007.

[11] Eva Kallin, Polynomial convexity: The three spheres problem, *Proceedings of the Conference in Complex Analysis (Minneapolis, 1964)*, pp. 301–304, Springer, Berlin, 1965.

constructed an example of three closed, pairwise disjoint polydiscs in $\mathbb{C}^3$ whose union is *not* polynomially convex.

Runge's theorem in dimension 1 indicates that polynomial convexity is intimately connected with the approximation of holomorphic functions by polynomials. There is an analogue of Runge's theorem in higher dimension, known as the Oka–Weil theorem. Here is the statement.

Theorem 10 (Oka–Weil)**.** *If K is a compact, polynomially convex set in $\mathbb{C}^n$, then every function holomorphic in a neighborhood of K can be approximated uniformly on K by (holomorphic) polynomials.*

Exercise 18. Give an example of a compact set K in $\mathbb{C}^2$ such that every function holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials, yet K is not polynomially convex.

3.2.2 Linear and rational convexity

The preceding examples involve functions that are globally defined on the whole space. But in many interesting cases, the class of functions depends on the region under consideration.

Suppose that G is an open set in $\mathbb{C}^n$, and let $\mathcal{F}$ be the class of those linear fractional functions

$$\frac{a_0 + a_1 z_1 + \cdots + a_n z_n}{b_0 + b_1 z_1 + \cdots + b_n z_n}$$

that happen to be holomorphic on G (in other words, the denominator has no zeroes inside G). Strictly speaking, one should write $\mathcal{F}_G$, but usually the open set G will be clear from context. By the solution of Exercise 15, every convex set is $\mathcal{F}$-convex. A simple example of a nonconvex but $\mathcal{F}$-convex open set is $\mathbb{C}^2 \setminus \{ (z_1, z_2) \in \mathbb{C}^2 : z_2 = 0 \}$: for if K is a compact subset of this open set, then $\widehat{K}_{\mathcal{F}}$ stays away from the boundary of the open set, since the function $1/z_2$ is bounded on K. The claim is that an open set G is $\mathcal{F}$-convex if and only if through each boundary point of G there passes a complex hyperplane that does not intersect G (a so-called supporting hyperplane).

To prove the claim, suppose first that G is $\mathcal{F}$-convex, and let w be a point in the boundary of G. If K is a compact subset of G, then $\widehat{K}_{\mathcal{F}}$ is again a compact subset of G, so to every point w' in G sufficiently close to w there corresponds a linear fractional function f in $\mathcal{F}$ such that $f(w') = 1 > \max\{ |f(z)| : z \in K \}$. Let ℓ denote the difference between the numerator of f and the denominator of f; then $\ell(z) = 0$ at a point z in G if and only if $f(z) = 1$. Hence the zero set of ℓ, which is a complex hyperplane, passes through w' and does not intersect K. Multiply ℓ by a suitable constant to ensure that the vector consisting of the coefficients of ℓ has length 1.

Now exhaust G by an increasing sequence $\{K_j\}$ of compact sets. The preceding construction produces a sequence $\{w_j\}$ of points in G converging to w and a sequence $\{\ell_j\}$ of normalized first-degree polynomials such that $\ell_j(w_j) = 0$, and the zero set of ℓ_j does not intersect K_j. The set of vectors of length 1 is compact, so it is possible to pass to the limit of a suitable subsequence to obtain a complex hyperplane that passes through the boundary point w and does not intersect the open set G.

Conversely, a supporting complex hyperplane at a boundary point w is the zero set of a certain first-degree polynomial ℓ, and $1/\ell$ is then a linear fractional function that is holomorphic on G and blows up at w. Therefore the $\mathcal{F}$-convex hull of a compact set K in G stays away from w. Since w is arbitrary, the hull $\widehat{K}_{\mathcal{F}}$ is a compact subset of G. Since K is arbitrary, the domain G is $\mathcal{F}$-convex by definition.

A domain is called *weakly linearly convex* if it is convex with respect to the linear fractional functions that are holomorphic on it. (A domain is called *linearly convex* if the complement can be written as a union of complex hyperplanes. The terminology is not completely standardized, however, so one has to check each author's definitions. There are examples of weakly linearly convex domains that are not linearly convex. The idea can be seen already in $\mathbb{R}^2$. Take an equilateral triangle of side length 1 and erase the middle portion, leaving in the corners three equilateral triangles of side length slightly less than $1/2$. There is a supporting line through each boundary point of this disconnected set, but there is no line through the origin that is disjoint from the three triangles. This idea can be implemented in $\mathbb{C}^2$ to construct a connected, weakly linearly convex domain that is not linearly convex.[12])

Next consider general rational functions (quotients of polynomials). A compact set K is called *rationally convex* if every point w outside K can be separated from K by a rational function that is holomorphic on $K \cup \{w\}$, that is, if there is a rational function f such that $|f(w)| > \max\{\,|f(z)| : z \in K\,\}$. In this definition, it does not much matter whether f is holomorphic at the point w, for if $f(w)$ is undefined, then one can slightly perturb the coefficients of f to make $|f(w)|$ be a large finite number without changing the values of f on K very much.

Example 7. Every compact set K in $\mathbb{C}$ is rationally convex. Indeed, if w is a point outside K, then the rational function $1/(z-w)$ blows up at w, so w is not in the rationally convex hull of K. [For a suitably small positive ϵ, the rational function $1/(z-w-\epsilon)$ has larger modulus at w than it does anywhere on K.]

There is a certain awkwardness in talking about multi-variable rational functions, because the singularities can be either poles (where the modulus blows up) or points of indeterminacy (like the origin for the function z_1/z_2). Therefore it is convenient to rephrase the notion of rational convexity in a way that uses only polynomials.

The notion of polynomial convexity involves separation by the modulus of a polynomial; it is natural to introduce the modulus in order to write inequalities. But one could consider the weaker separation property that a point w is separated from a compact set K if there is a polynomial p such that the image of w under p is not contained in the image of K under p. The claim is that this weaker separation property is identical to the notion of rational convexity.

Indeed, if the point $p(w)$ is not in the set $p(K)$, then for every sufficiently small positive ϵ, the function $1/(p(z) - p(w) - \epsilon)$ is a rational function of z that is holomorphic in a neighborhood of K and has larger modulus at w than it has anywhere on K. Conversely, if f is a rational

[12] A reference is Mats Andersson, Mikael Passare, and Ragnar Sigurdsson, *Complex Convexity and Analytic Functionals*, Birkhäuser, 2004, Example 2.1.7.

function, holomorphic on $K \cup \{w\}$, whose modulus separates w from K, then the function $1/(f(z) - f(w))$ is a rational function of z that is holomorphic on K and singular at w; this function can be rewritten as a quotient of polynomials, the denominator being a polynomial that is equal to 0 at w and is nonzero on K.

Thus, a point w is in the rationally convex hull of a compact set K if and only if every polynomial that is equal to zero at w also has a zero on K.

Exercise 19. The rationally convex hull of a compact subset of $\mathbb{C}^n$ is again a compact subset of $\mathbb{C}^n$.

Example 8 (the Hartogs triangle). The open set $\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1 \}$ is convex with respect to the linear fractional functions, because through each boundary point passes a complex line that does not intersect the domain. Indeed, the line where $z_2 = 0$ serves at the origin $(0, 0)$; at any other boundary point where the two coordinates have equal modulus, there is some value of θ for which a suitable line is the one that sends the complex paramater λ to $(\lambda, e^{i\theta}\lambda)$; and at a boundary point where the second coordinate has modulus equal to 1, there is some value of θ for which the line where $z_2 = e^{i\theta}$ serves.

In particular, the open Hartogs triangle is a rationally convex domain, since there are more rational functions than there are linear fractions. On the other hand, the open Hartogs triangle is not polynomially convex. Indeed, consider the circle $\{ (0, \frac{1}{2}e^{i\theta}) : 0 \le \theta < 2\pi \}$: no point of the disc bounded by this circle can be separated from the circle by a polynomial, so the polynomial hull of the circle with respect to the open Hartogs triangle is not a compact subset of the triangle.

Next consider the *closed* Hartogs triangle, the set where $|z_1| \le |z_2| \le 1$. The rationally convex hull of this compact set is the whole closed bidisc. Indeed, suppose p is a polynomial that has no zero on the closed Hartogs triangle; by continuity, p has no zero in an open neighborhood of the closed triangle. Consequently, the reciprocal $1/p$ is holomorphic in a Hartogs figure, so by Theorem 4, the function $1/p$ extends to be holomorphic on the whole (closed) bidisc. Therefore the polynomial p cannot have any zeroes in the bidisc. By the characterization of rational convexity in terms of zeroes of polynomials, it follows that the rational hull of the closed Hartogs triangle contains the whole bidisc; the rational hull cannot contain any other points, since the rational hull is a subset of the convex hull.