# Math 685 Notes
# Topics in Several Complex Variables

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# Contents

# 1 Integral Representations

Cauchy's integral formula is the driving force in the elementary theory of one-dimensional complex analysis. The following review of Cauchy's formula is a springboard for the study of integral representations in higher dimension.

## 1.1 Cauchy's formula with remainder in $\mathbb{C}$

Suppose that $\Omega$ is a bounded domain in $\mathbb{C}$ with class $C^1$ boundary $\gamma$. (The boundary $\gamma$ is not necessarily connected: it may consist of several closed curves.) Suppose that $f$ is a class $C^1$ function on the closure of $\Omega$. Then

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}\,dw - \frac{1}{\pi}\iint_\Omega \frac{\partial f/\partial \overline{w}}{w-z}\,d\mathsf{Area}_w = \begin{cases} f(z), & z \in \Omega, \\ 0, & z \notin \text{closure}(\Omega), \end{cases} \tag{1.1}$$

where the second integral is an absolutely convergent improper integral.

The proof is a standard calculation using Green's theorem in complex form, which says that if $g$ is a class $C^1$ function on the closure of $\Omega$, then

$$\oint_\gamma g(w)\,dw = 2i\int_\Omega \frac{\partial g}{\partial \overline{w}}\,d\mathsf{Area}_w.$$

If $z$ is outside the closure of $\Omega$, then $1/(w-z)$ has no singularity for $w$ inside $\Omega$, so the second case of (1.1) follows immediately from this version of Green's theorem. When $z \in \Omega$, Green's theorem is not directly applicable to the first integral in (1.1), because the integrand has a singularity inside $\Omega$. One overcomes this difficulty by adding and subtracting the integral over the boundary of a small disc $B(z,\epsilon)$ centered at $z$ with radius $\epsilon$:

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\oint_{bB(z,\epsilon)} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\left(\oint_\gamma - \oint_{bB(z,\epsilon)}\right)\frac{f(w)}{w-z}\,dw \\ &\overset{\text{Green}}{=} \frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})\,d\theta + \frac{1}{\pi}\iint_{\Omega\setminus B(z,\epsilon)} \frac{\partial f/\partial \overline{w}}{w-z}\,d\mathsf{Area}_w. \end{aligned} \tag{1.2}$$

When $\epsilon \to 0$, the first integral in (1.2) approaches the limit $f(z)$, and the second integral turns into the (convergent, improper) area integral in (1.1).

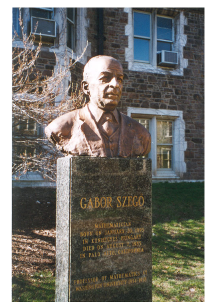
## Specializations

If the function $f$ in Cauchy's formula (1.1) is assumed additionally to be holomorphic, then the formula reduces to the familiar version that represents a holomorphic function in a domain by an integral of its boundary values:

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z}\, dw, \qquad z \in \Omega. \tag{1.3}$$

By specializing to the case of the unit disc, one can easily rewrite this formula in a different way. Using that $w \cdot \overline{w} = 1$ on the boundary of the unit disc and that $\overline{w}\, dw = i\, d\sigma_w$ (where $d\sigma_w$ is the arc-length element on the boundary of the unit disc), one finds that

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{\overline{w}(w - z)}\, \overline{w}\, dw = \frac{1}{2\pi} \int_\gamma \frac{f(w)}{1 - z\overline{w}}\, d\sigma. \tag{1.4}$$



Gábor Szegő
(1895–1985)

The expression $\frac{1}{2\pi} \cdot \frac{1}{1 - z\overline{w}}$ is the *Szegő reproducing kernel function for the unit disc*.

In (1.4), the singularity of the integrand inside the unit disc $D$ has evaporated. Consequently, one can rewrite the formula via Green's theorem as an area integral:

Stefan Bergman
(1895–1977)

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{\overline{w}(w - z)}\, \overline{w}\, dw = \frac{1}{\pi} \int_D f(w) \frac{\partial}{\partial \overline{w}} \left[ \frac{\overline{w}}{1 - z\overline{w}} \right] d\mathsf{Area}_w \\ &= \frac{1}{\pi} \int_D \frac{f(w)}{(1 - z\overline{w})^2}\, d\mathsf{Area}_w. \end{aligned}$$

The expression $\frac{1}{\pi} \cdot \frac{1}{(1 - z\overline{w})^2}$ is the *Bergman reproducing kernel function for the unit disc*.

# 1.2 Background on differential forms

The language of differential forms is essential for discussing integral representations in dimensions greater than 1. There is no lack of systematic developments of differential forms, ranging from Spivak's little book [34] to Cartan's course [6] to Lee's graduate text [21, Chapters 12–14]. For differential forms in the setting of complex analysis, one may consult, for example, Range's book [29, Chapter III].

The aim of this section is to give a brief, unsystematic discussion of differential forms that gives enough understanding of the notation to make it possible to proceed with the complex analysis. The first goal is to attach a meaning to the symbol $dx_j$.

One point of view is that $dx_j$ is merely a formal symbol used as a placeholder in integration. For example, if $\gamma\colon [0,1] \to \mathbb{R}^2$ is a curve with components $\gamma_1$ and $\gamma_2$, then the integral

$$\int_\gamma P(x_1,x_2)\,dx_1 + Q(x_1,x_2)\,dx_2 \tag{1.5}$$

can be considered an abbreviation for the ordinary calculus integral

$$\int_0^1 \{P(\gamma_1(t),\gamma_2(t))\gamma_1'(t) + Q(\gamma_1(t),\gamma_2(t))\gamma_2'(t)\}\;dt. \tag{1.6}$$

The other point of view is that $dx_j$ is an honest mathematical object. The differential operator $\partial/\partial x_j$ may be thought of as a basis element in a vector space (the space of "tangent vectors", which can be defined independently of coordinates as a space of equivalence classes of tangent curves), and $dx_j$ may be viewed as the dual basis element in the dual vector space. From this point of view, the "differential form" $P\,dx_1+Q\,dx_2$ exists as an element of a certain vector space, and the expression (1.6) serves to define the integral of a differential form along a curve. The intuitive idea is that to integrate a differential form on a curve (more generally, on a manifold), one "pulls back" the form to a flat coordinate system, and in the flat coordinates one knows what integration should mean. The formula from one-dimensional calculus about changing variables in an integral ("method of substitution") shows that the value of the integral along a curve is independent of the choice of coordinates, as long as one preserves the orientation of the curve.

In the case of integration of a differential form of degree 1, the second interpretation of the symbol $dx_j$ seems rather pedantic. In the case of integration over higher-dimensional sets, however, the notation of differential forms is an essential tool.

The motivation for the definition of the "wedge product" of differential forms is the formula from calculus for changing variables in multiple integrals. If $\Omega$ is a domain in $\mathbb{R}^n$, and $T\colon \Omega \to \mathbb{R}^n$ is an invertible coordinate transformation, then

$$\int_{T(\Omega)} f(y_1,\dots,y_n)\,dy_1\dots dy_n = \int_\Omega (f\circ T)(x)\,|\det DT|\,dx_1\dots dx_n,$$

where $DT$ is the derivative (Jacobian matrix) of $T$. The determinant is an alternating, multilinear function of its rows, and this property underlies the definition of the product of differential forms.

Viewed as formal symbols, differential forms of degree $k$ are linear combinations of expressions $dx_{j_1}\wedge\cdots\wedge dx_{j_k}$ subject to the rules that $dx_j\wedge dx_j = 0$ and $dx_j\wedge dx_k = -dx_k\wedge dx_j$.

Viewed as honest mathematical objects, differential $k$-forms are alternating, multilinear forms on the Cartesian product of $k$ copies of the tangent space.

To integrate a differential $k$-form on a $k$-dimensional manifold, one uses a partition of unity (if necessary) to reduce to a problem in a single coordinate patch. Then one pulls back the form to get an integral in flat coordinates, where one knows how to integrate after "forgetting the wedges". A significant technical point is that the orientation of the local coordinates must be compatible with the orientation of the manifold.

**Exercise 1.2.1.** Let $\Gamma$ denote the "upper" half of the unit sphere in $\mathbb{R}^4$, that is, the set of points $(x_1, x_2, x_3, x_4)$ such that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ and $x_4 > 0$. Orient $\Gamma$ such that the normal vector to the surface points "up" (that is, has a positive component in the direction of increasing $x_4$). Evaluate the integral

$$\int_\Gamma 3x_1x_4\, dx_2 \wedge dx_3 \wedge dx_4 - x_4^2\, dx_1 \wedge dx_2 \wedge dx_3 \tag{1.7}$$

(a) by parametrizing the surface, and

(b) by applying Stokes's theorem (that is, $\int_{b\Omega} \omega = \int_\Omega d\omega$).

Answer: $4\pi/3$.

In the preceding exercise, the form $dx_1 \wedge dx_2 \wedge dx_3$ gives a *negative* orientation to $\Gamma$. If a surface in $\mathbb{R}^n$ is the zero set of a function $\rho$ and is oriented by the gradient vector $\nabla\rho$, then (according to the standard convention) a form $\omega$ of degree $n-1$ on this surface is compatibly oriented if the $n$-form $d\rho \wedge \omega$ has the same sign as the volume form in $\mathbb{R}^n$.

**Exercise 1.2.2.** Suppose that $\Omega$ is a domain in $\mathbb{R}^n$, and $\rho$ is a class $C^1$ function such that $\Omega = \{\, x : \rho(x) < 0 \,\}$ and $b\Omega = \{\, x : \rho(x) = 0\}$ and $\nabla\rho \neq 0$ on $b\Omega$. Orient $b\Omega$ compatibly with $\nabla\rho$. Writing

$$dx[j] := dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n, \tag{1.8}$$

show that for every continuous function $f$ on $b\Omega$ one has

$$\int_{b\Omega} f(x)\, dx[j] = \int_{b\Omega} f(x)(-1)^{j-1} \frac{\partial\rho/\partial x_j}{|\nabla\rho(x)|}\, d\mathsf{SurfaceArea}. \tag{1.9}$$

Note that $d\mathsf{SurfaceArea}$ on the right-hand side represents the surface area measure on the boundary in the sense of measure theory: it is not a differential form.

The standard orientation on $\mathbb{R}^n$ is $dx_1 \wedge \cdots \wedge dx_n$, but there is no canonical orientation on $\mathbb{C}^n$. From now on, the assumption will be that $\mathbb{C}^n$ is oriented by the volume form $dV$ such that

$$dV = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n. \tag{1.10}$$

This orientation agrees with the one used in the books by Krantz [18, p. 2], Range [29, p. 133], and Rudin [32, pp. 335 and 344], but it differs from the orientation used in Kytmanov's book [19, p. 1]. Kytmanov takes the volume form to be $dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n$, which differs from (1.10) by the factor $(-1)^{n(n-1)/2}$.

**Exercise 1.2.3.** Verify the following relations.

$$dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n = (-2i)^n\, dV \tag{1.11}$$

$$dz \wedge d\overline{z} := dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n = (-1)^{n(n-1)/2}(-2i)^n\, dV \tag{1.12}$$

$$d\overline{z} \wedge dz := d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n = (-1)^{n(n-1)/2}(2i)^n\, dV \tag{1.13}$$

$$(dz_1 \wedge d\overline{z}_1 + \cdots + dz_n \wedge d\overline{z}_n)^n = n!\,(-2i)^n\, dV \tag{1.14}$$

## 1.3 The Bochner–Martinelli kernel

One useful feature of the Cauchy integral in one-dimensional complex analysis is that the kernel is universal: it does not depend on the domain $\Omega$. When $n \geq 2$, one can write in $\mathbb{C}^n$ an iterated Cauchy integral, which too has a universal kernel, but the integration is over a set of real dimension $n$, not over the whole $(2n-1)$-dimensional boundary of a domain. The Bochner–Martinelli integral is a universal representation formula for holomorphic functions via integration over the whole boundary of a domain.

**Definition 1.3.1.** The Bochner–Martinelli kernel $U(w, z)$ in $\mathbb{C}^n$ is a differential form of bidegree $(n, n-1)$ given by the expression

$$c_n \sum_{j=1}^{n} \frac{(-1)^{j-1}(\overline{w}_j - \overline{z}_j)}{|w - z|^{2n}}\, d\overline{w}[j] \wedge dw, \tag{1.15}$$

where the dimensional constant $c_n := (-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n$, the $(0, n-1)$-form $d\overline{w}[j] := d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_{j-1} \wedge d\overline{w}_{j+1} \wedge \cdots \wedge d\overline{w}_n$, and the $(n, 0)$-form $dw := dw_1 \wedge \cdots \wedge dw_n$. The sign of $c_n$ reflects the choice of orientation of $\mathbb{C}^n$.

When $n = 1$, the Bochner–Martinelli kernel evidently reduces to the Cauchy kernel $\frac{1}{2\pi i} \cdot \frac{1}{w-z}\, dw$. When $n \geq 2$, however, the Bochner–Martinelli kernel differs significantly from the iterated Cauchy kernel. First of all, the iterated Cauchy kernel is an $(n, 0)$-form, while the Bochner–Martinelli kernel is an $(n, n-1)$-form, so the two kernels are integrated over manifolds of different dimensions. Secondly, the Bochner–Martinelli kernel (1.15) is *not holomorphic* in the free variable $z$. Nonetheless, the Bochner–Martinelli kernel is useful in the study of holomorphic functions because of the following key property:

$$d_w U(w,z) = \overline{\partial}_w U(w,z) = 0 \qquad \text{when } w \neq z. \tag{1.16}$$

The first equality holds trivially by degree considerations. The following straightforward computation shows that $U(w,z)$ is $\overline{\partial}_w$-closed:

$$\begin{aligned}
\overline{\partial}_w U(w,z) &= c_n \sum_{j=1}^{n} \frac{\partial}{\partial \overline{w}_j} \left( (\overline{w}_j - \overline{z}_j)(|w-z|^2)^{-n} \right) \, d\overline{w} \wedge dw \\
&= c_n \sum_{j=1}^{n} \left( (|w-z|^2)^{-n} - n(\overline{w}_j - \overline{z}_j)(w_j - z_j)(|w-z|^2)^{-(n+1)} \right) \, d\overline{w} \wedge dw \\
&= c_n \left( \frac{n}{|w-z|^{2n}} - n \frac{|w-z|^2}{|w-z|^{2n+2}} \right) d\overline{w} \wedge dw = 0.
\end{aligned}$$

An alternate expression for the Bochner–Martinelli kernel (1.15) is illuminating:

$$U(w,z) = \frac{(-1)^n c_n}{n-1} \, \partial \left( \frac{1}{|w-z|^{2n-2}} \right) \wedge \sum_{j=1}^{n} d\overline{w}[j] \wedge dw[j] \qquad \text{when } n \geq 2, \tag{1.17}$$

as a routine calculation shows. Since $|w-z|^{-(2n-2)}$ is (a constant times) the fundamental solution for the Laplace operator $\Delta$ in $\mathbb{C}^n$ (viewed as $\mathbb{R}^{2n}$), the expression (1.17) shows that the coefficients of the Bochner–Martinelli kernel are *harmonic* functions of $z$.

**Lemma 1.3.2.** *If $r > 0$, then $\int_{w \in bB(z,r)} U(w,z) = 1$, independently of $r$.*

*Proof.* Let $\xi$ be a new variable in $\mathbb{C}^n$ such that $w = z + r\xi$. Then $U(w,z) = U(r\xi, 0) = c_n \sum_{j=1}^{n} \frac{(-1)^j \overline{\xi}_j}{|\xi|^{2n}} \, d\overline{\xi}[j] \wedge d\xi$. By Stokes's theorem,

$$\int_{w \in bB(z,r)} U(w,z) = c_n \int_{\xi \in bB(0,1)} \sum_{j=1}^{n} (-1)^j \overline{\xi}_j \, d\overline{\xi}[j] \wedge d\xi = c_n \int_{B(0,1)} n \, d\overline{\xi} \wedge d\xi.$$

The volume of the unit ball in $\mathbb{C}^n$ is equal to $\pi^n/n!$ (see the following exercise), so by equation (1.13), the integral on the right-hand side equals $nc_n(-1)^{n(n-1)/2}(2i)^n \pi^n/n!$, which reduces to 1 by the choice of $c_n$. $\square$

**Exercise 1.3.3.** Show that the surface area of the unit ball in $\mathbb{C}^n$ is equal to $2\pi^n/(n-1)!$, and the volume of the unit ball in $\mathbb{C}^n$ is equal to $\pi^n/n!$.
*Hint*: To calculate the surface area $\sigma_n$, compute $\int_{\mathbb{C}^n} e^{-|z|^2}\,dV_z$ first in rectangular coordinates as $\left(\int_{\mathbb{C}} e^{-(x^2+y^2)}\,dx\,dy\right)^n$ and then in spherical coordinates as $\int_0^\infty \sigma_n r^{2n-1}e^{-r^2}\,dr$, and compare the answers. To calculate the volume, integrate in spherical coordinates using the expression for $\sigma_n$. For more about this trick, see [7] and its references.

**Theorem 1.3.4.** *Suppose that $\Omega$ is a bounded domain in $\mathbb{C}^n$ with class $C^1$ boundary, and $f$ is a function of class $C^1$ on the closure of $\Omega$. Then*

$$\int_{b\Omega} f(w)\,U(w,z) - \int_{\Omega} \overline{\partial} f(w)\wedge U(w,z) = \begin{cases} f(z), & z\in\Omega,\\ 0, & z\notin \text{closure}(\Omega). \end{cases} \tag{1.18}$$

The second integral in (1.18) is an absolutely convergent improper integral, because when $w$ approaches $z$, the Bochner–Martinelli kernel has a singularity of order $|w-z|^{-(2n-1)}$.

**Corollary 1.3.5** (Bochner–Martinelli formula)**.** *If additionally $f$ is holomorphic in $\Omega$, then*

$$\int_{b\Omega} f(w)\,U(w,z) = \begin{cases} f(z), & z\in\Omega,\\ 0, & z\notin \text{closure}(\Omega). \end{cases} \tag{1.19}$$

*Proof of Theorem 1.3.4.* The idea is the same as in the proof of Cauchy's formula with remainder. Because $d_w U(w,z) = 0$ when $w\neq z$, and $U(w,z)$ has bidegree $(n, n-1)$, one has that $d_w(f(w)U(w,z)) = \overline{\partial} f(w)\wedge U(w,z)$. If $z$ is outside the closure of $\Omega$, then $U(w,z)$ has no singularity for $w$ inside $\Omega$, and the second case of (1.18) follows immediately from Stokes's theorem.

When $z\in\Omega$, one should add and subtract the integral over a ball $B(z,\epsilon)$ centered at $z$ with radius $\epsilon$, apply Stokes's formula, add and subtract $f(z)$ in the integrand of the boundary integral, and take the limit as $\epsilon\to 0$. Thus

$$\begin{aligned} \int_{b\Omega} f(w)\,U(w,z) &= \int_{bB(z,\epsilon)} f(w)\,U(w,z) + \left(\int_{b\Omega} - \int_{bB(z,\epsilon)}\right) f(w)\,U(w,z) \\ &\overset{\text{Stokes}}{=} \int_{bB(z,\epsilon)} (f(w)-f(z))\,U(w,z) + \int_{bB(z,\epsilon)} f(z)\,U(w,z) + \int_{\Omega\setminus B(z,\epsilon)} \overline{\partial} f(w)\wedge U(w,z). \end{aligned} \tag{1.20}$$

When $w\in bB(z,\epsilon)$, the coefficients of $U(w,z)$ are of order $\epsilon^{-(2n-1)}$, while the surface area of $bB(z,\epsilon)$ is of order $\epsilon^{2n-1}$. Since $|f(w)-f(z)| = O(|w-z|) = O(\epsilon)$ when $w\in bB(z,\epsilon)$, the first integral in the second line of (1.20) tends to 0 with $\epsilon$. By Lemma 1.3.2, the second integral equals $f(z)$. The third term tends in the limit to the (convergent, improper) volume integral in (1.18). □

### 1.3.1 Example: the unit ball

Although the Bochner–Martinelli kernel does not depend on the domain when the kernel is displayed as a differential form, the expression of the kernel does change with the domain if the kernel is written as a function times surface area measure. Using Exercises 1.2.2 and 1.2.3, one can compute that if a domain $\Omega$ has defining function $\rho$, then

$$d\overline{w}[j] \wedge dw|_{b\Omega} = (-1)^{n(n-1)/2}(2i)^n(-1)^{j-1}\frac{\partial\rho/\partial\overline{w}_j}{|\nabla\rho|}\, d\mathsf{SurfaceArea}_w. \tag{1.21}$$

Consequently, the definition (1.15) can be rewritten as follows in terms of the function $\rho$ defining the boundary:

$$U(w,z) = \frac{(n-1)!}{\pi^n}\sum_{j=1}^{n}\frac{\partial\rho/\partial\overline{w}_j}{|\nabla\rho|}\cdot\frac{\overline{w}_j-\overline{z}_j}{|w-z|^{2n}}\, d\mathsf{SurfaceArea}_w. \tag{1.22}$$

A natural choice for the defining function $\rho$ of the unit ball in $\mathbb{C}^n$ is $\sum_{j=1}^{n}|w_j|^2-1$. Then $|\nabla\rho| = 2$ on the boundary of the unit ball, and $\partial\rho/\partial\overline{w}_j = w_j$. Writing $\langle w,z\rangle$ for $\sum_{j=1}^{n} w_j\overline{z}_j$ and using that $\langle w,w\rangle = 1$ on the boundary of the ball gives that

$$U(w,z) = \frac{(n-1)!}{2\pi^n}\cdot\frac{1-\langle w,z\rangle}{|w-z|^{2n}}\, d\mathsf{SurfaceArea}_w \qquad \text{for the unit ball.} \tag{1.23}$$

Since the Bochner–Martinelli kernel reproduces the constant function 1 (in particular), evaluating for $z$ equal to 0 shows that the constant $(n-1)!/2\pi^n$ is the reciprocal of the surface area of the unit ball (compare Exercise 1.3.3). It is interesting to compare (1.23) with the expression for the Poisson kernel $P(w,z)$ for the unit ball:

$$P(w,z) = \frac{(n-1)!}{2\pi^n}\cdot\frac{1-|z|^2}{|w-z|^{2n}}\, d\mathsf{SurfaceArea}_w. \tag{1.24}$$

### 1.3.2 First application: basic properties of holomorphic functions

Often one begins a course in complex function theory by defining functions to be holomorphic if they are of class $C^1$ and satisfy the Cauchy–Riemann equations. Then one needs to show that such functions automatically are class $C^\infty$. One can prove this property by using the iterated Cauchy integral, but the Bochner–Martinelli integral representation (1.19) serves equally well. One can differentiate under the integral sign in (1.19) to see that holomorphic functions are class $C^\infty$.

Next one would like to have estimates of Cauchy type for the derivatives of holomorphic functions. In (1.22), the kernel is $O(r^{-(2n-1)})$ for $w$ in $bB(z,r)$, and each differentiation with respect to $z$ worsens this estimate by one power of $r$. Therefore one sees immediately that for each multi-index $\alpha$ there is a constant $C_\alpha$ such that if $M$ is an upper bound for $|f(w)|$ on $B(z,r)$, then

$$|\partial^\alpha f(z)| \le C_\alpha M/r^{|\alpha|}.$$

The preceding properties of holomorphic functions drive the elementary theory: for instance, the existence of convergent power series expansions, Liouville's theorem, and arguments about normal families.

### 1.3.3 Second application: solving $\overline{\partial}$ with compact support

One of the important features of complex analysis in $\mathbb{C}^n$ that holds when $n \ge 2$ but not when $n = 1$ is the solvability of the inhomogeneous Cauchy–Riemann equations with *compact support*. Since this result depends on the dimension, it is part of the "non-elementary" theory of complex analysis. One can use the Bochner–Martinelli kernel to give an explicit formula for the solution of the $\overline{\partial}$-problem with compact support.

**Theorem 1.3.6.** *Suppose $n \ge 2$, and $g$ is a compactly supported $(0,1)$-form in $\mathbb{C}^n$ of class $C^1$ that is $\overline{\partial}$-closed: namely, if $g(z) = \sum_{j=1}^n g_j(z)\, d\overline{z}_j$, then $\partial g_j/\partial \overline{z}_k = \partial g_k/\partial \overline{z}_j$ for all $j$ and $k$. If*

$$f(z) := -\int_{\mathbb{C}^n} g(w) \wedge U(w,z),$$

*then $f$ is a compactly supported function such that $\overline{\partial} f = g$.*

*Proof.* Introduce a new variable $\xi$ equal to the difference $w - z$. Then

$$f(z) = -\int_{\mathbb{C}^n} \sum_{j=1}^n g_j(\xi + z)\, d\overline{\xi}_j \wedge U(\xi, 0).$$

Differentiate under the integral sign, using that

$$\frac{\partial}{\partial \overline{z}_k}(g_j(\xi + z)) = \frac{\partial}{\partial \overline{\xi}_k}(g_j(\xi + z)),$$

and then change variables back again to get that

$$\frac{\partial f}{\partial \overline{z}_k}(z) = -\int_{\mathbb{C}^n} \sum_{j=1}^n \frac{\partial g_j}{\partial \overline{w}_k}(w)\, d\overline{w}_j \wedge U(w,z).$$

Use the hypothesis that $\partial g_j/\partial \overline{w}_k = \partial g_k/\partial \overline{w}_j$ to obtain that

$$\frac{\partial f}{\partial \overline{z}_k}(z) = -\int_{\mathbb{C}^n} \overline{\partial} g_k \wedge U(w,z).$$

On the other hand, applying Theorem 1.3.4 to $g_k$ on a large ball containing both the support of $g$ (so that the boundary integral in (1.18) vanishes) and the point $z$ yields that

$$g_k(z) = -\int_{\mathbb{C}^n} \overline{\partial} g_k(w) \wedge U(w,z).$$

Comparing the preceding two formulas shows that $\partial f/\partial \overline{z}_k = g_k$ for an arbitrary $k$. Hence $\overline{\partial} f = g$, as claimed.

It remains to show that $f$ has compact support. This part of the argument uses in a crucial way that $n \geq 2$. Since $\overline{\partial} f = g$, the function $f$ is holomorphic on the complement of the support of $g$. Let $R$ be a positive real number such that the ball $B(0,R)$ contains the support of $g$, and fix values of $z_2, \ldots, z_n$ such that $\min(|z_2|, \ldots, |z_n|) > R$. Then $f(z_1, z_2, \ldots, z_n)$ is an entire holomorphic function of $z_1$. It is evident from the explicit form of the Bochner–Martinelli kernel (1.15) and the definition of $f$ that $f(z_1, z_2, \ldots, z_n) \to 0$ when $|z_1| \to \infty$. The only entire function of $z_1$ that tends to 0 at infinity is the 0 function. Thus $f(z_1, z_2, \ldots, z_n) = 0$ when $\min(|z_2|, \ldots, |z_n|) > R$. Since $f$ is holomorphic on the complement of the support of $g$, it follows that $f$ is identically equal to 0 on the unbounded component of the complement of the support of $g$. Thus $f$ has compact support. $\square$

Once one knows how to solve the $\overline{\partial}$-equation with compact support, one can prove a version of the theorem of Hartogs about continuation of holomorphic functions across compact holes. See, for example, Hörmander's book [10, pp. 30–31]. One can, however, prove the theorem of Hartogs directly and explicitly from the Bochner–Martinelli integral representation, and this is the next topic.

### 1.3.4 The Hartogs phenomenon

One version of the theorem of Hartogs is the following.

**Theorem 1.3.7.** *Suppose $n \geq 2$. Let $K$ be a compact subset of a bounded domain $\Omega$ in $\mathbb{C}^n$ such that $\Omega \setminus K$ is connected. The every function that is holomorphic on $\Omega \setminus K$ is the restriction to $\Omega \setminus K$ of a function that is holomorphic on $\Omega$.*

*Proof.* Suppose $f$ is a holomorphic function on $\Omega \setminus K$. By shrinking $\Omega$, one can assume without loss of generality that $\Omega$ has class $C^1$ boundary and that $f$ is holomorphic in a neighborhood of $b\Omega$ in $\mathbb{C}^n$. If there is a holomorphic extension $F$ of $f$ into all of $\Omega$, then that extension is represented by the Bochner–Martinelli formula (1.19), so there is a natural way to define a candidate extension: namely,

$$F(z) := \int_{b\Omega} f(w)\, U(w,z), \qquad z \notin b\Omega.$$

Two properties need to be verified: first, that $F$ is holomorphic in $\Omega$, and second, that $F$ agrees with $f$ on some open subset of $\Omega \setminus K$.

Verifying the first property requires a key identity, which says that the derivative of the Bochner–Martinelli kernel $U(w,z)$ with respect to $\overline{z}_k$ is $\overline{\partial}_w$-exact.

**Lemma 1.3.8.** *When $w \neq z$, the Bochner–Martinelli kernel $U(w,z)$ satisfies for each $k$ the following property:*

$$\frac{\partial}{\partial \overline{z}_k} U(w,z) = \overline{\partial}_w V_k(w,z),$$

*where*

$$V_k(w,z) := c_n \sum_{j\neq k} (-1)^{j+k+\epsilon(j,k)} \frac{\overline{w}_j - \overline{z}_j}{|w-z|^{2n}}\, d\overline{w}[j,k] \wedge dw.$$

*The notation $d\overline{w}[j,k]$ means that $d\overline{w}_j$ and $d\overline{w}_k$ have been omitted from the wedge product (the other differentials being in increasing order), and the symbol $\epsilon(j,k)$ equals $0$ if $j<k$ and $1$ if $j>k$. The constant $c_n$ is the same as in Definition 1.3.1.*

Accepting this identity for the moment, observe that

$$\frac{\partial F}{\partial \overline{z}_k} = \int_{b\Omega} f(w)\, \overline{\partial}_w V_k(w,z) = \int_{b\Omega} \overline{\partial}_w (f(w)\, V_k(w,z))$$

because $f$ is holomorphic in a neighborhood of $b\Omega$. Since the form $V_k$ has bidegree $(n, n-2)$, one has that $\overline{\partial}_w(f(w)\, V_k(w,z)) = d_w(f(w)\, V_k(w,z))$. Therefore $\partial F/\partial \overline{z}_k$, being the integral of an exact form over $b\Omega$, is equal to 0. Thus $F$ is holomorphic where it is defined: namely, both in the interior of $\Omega$ and in the complement of the closure of $\Omega$.

Moreover, the same argument as at the end of the preceding section shows that $F$ is identically equal to 0 in the unbounded component of the complement of the closure of $\Omega$: namely, the explicit form of the Bochner–Martinelli kernel shows that $F(z_1,\dots,z_n) \to 0$ when $|z_1| \to \infty$, and when $\min(|z_2|,\dots,|z_n|)$ is sufficiently large, the function $F(z_1,\dots,z_n)$ is holomorphic on the entire $z_1$ plane.

Let $\Omega'$ denote a relatively compact subdomain of $\Omega$ such that $\Omega \setminus \text{closure}(\Omega')$ is connected, $K \subset \Omega'$, and $\Omega'$ has class $C^1$ boundary. The preceding argument shows that $\int_{b\Omega'} f(z)\, U(w,z)$ is identically equal to 0 when $z$ is in the unbounded component of the complement of the closure of $\Omega'$, in particular, when $z \in \Omega \setminus \text{closure}(\Omega')$. On the other hand, applying the Bochner–Martinelli representation (1.18) to the domain $\Omega \setminus \text{closure}(\Omega')$ shows that

$$f(z) = \int_{b\Omega} f(w)\, U(w,z) - \int_{b\Omega'} f(w)\, U(w,z), \qquad z \in \Omega \setminus \text{closure}(\Omega'),$$

since $\overline{\partial} f = 0$ in $\Omega \setminus \Omega'$. The first term on the right-hand side is equal to $F(z)$, and the second term—as just observed—is equal to 0. Therefore the holomorphic functions $f$ and $F$ are equal on an open subset of the connected open set $\Omega \setminus K$, and hence on all of $\Omega \setminus K$. Thus $F$ is, as claimed, a holomorphic extension of $f$ from $\Omega \setminus K$ to $\Omega$.

It remains to prove Lemma 1.3.8. The essence of the lemma is the following routine calculation for harmonic functions, which may be interpreted as saying that the difference of two particular $(n-1, n-1)$ forms is $\partial$-closed.

**Exercise 1.3.9.** If $g$ is a harmonic function, then for each $k$ one has that

$$\partial\left(\frac{\partial g}{\partial \overline{w}_k}\right) \wedge \sum_{j=1}^{n} d\overline{w}[j] \wedge dw[j] = (-1)^k \partial\overline{\partial} g(w) \wedge \sum_{j \neq k} (-1)^{\epsilon(j,k)} d\overline{w}[j,k] \wedge dw[j].$$

*Hint*: Harmonicity means that $\displaystyle\sum_{j \neq k} \frac{\partial^2 g}{\partial w_j \partial \overline{w}_j} = -\frac{\partial^2 g}{\partial w_k \partial \overline{w}_k}$.

To apply the exercise, differentiate representation (1.17) for the Bochner–Martinelli kernel with respect to $\overline{z}_k$, observe that $\partial|w-z|^2/\partial\overline{z}_k = -\partial|w-z|^2/\partial\overline{w}_k$, and set $g(w)$ equal to the function $1/|w-z|^{2n-2}$ (which, as previously observed, is harmonic when $w \neq z$). Since $\partial U/\partial \overline{z}_k$ equals the left-hand side of the exercise multiplied by the constant $(-1)^{n+1} c_n/(n-1)$, and $\partial\overline{\partial} = -\overline{\partial}\partial$, the exercise implies that $\partial U/\partial\overline{z}_k$ is equal to $\overline{\partial}_w$ of the form

$$(-1)^{n+k} \frac{c_n}{n-1}\, \partial g(w) \wedge \sum_{j \neq k} (-1)^{\epsilon(j,k)}\, d\overline{w}[j,k] \wedge dw[j].$$

Since $\partial g(w) = \sum_{j=1}^{n} (-1)(n-1)((\overline{w}_j - \overline{z}_j)/|w-z|^{2n})\, dw_j$, the preceding quantity is equal to the expression given for $V_k(w,z)$ in the statement of Lemma 1.3.8. □

### 1.3.5 Functions reproduced by the Bochner–Martinelli integral

The Bochner–Martinelli formula (1.19) says that holomorphic functions in a domain are reproduced by integration against the Bochner–Martinelli kernel on the boundary. Are there any other functions that the Bochner–Martinelli integral reproduces? (The Poisson integral, for instance, reproduces not only holomorphic functions but also harmonic functions.) The answer to the question is negative, and this observation indicates that the Bochner–Martinelli kernel is in some sense the "right" kernel for studying holomorphic functions (although not necessarily the *only* right kernel).

**Theorem 1.3.10.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with class $C^1$ boundary. Suppose $f$ is a function of class $C^1$ on the closure of $\Omega$ such that*

$$f(z) = \int_{b\Omega} f(w)\, U(w,z) \qquad \text{for all } z \in \Omega.$$

*Then $f$ is holomorphic in $\Omega$.*

*Proof.* The goal is to show that $\overline{\partial} f = 0$ in $\Omega$. In view of version (1.18) of the Bochner–Martinelli formula with remainder, the hypothesis of the theorem implies that

$$\int_\Omega \overline{\partial} f(w) \wedge U(w,z) = 0, \qquad z \in \Omega. \tag{1.25}$$

Thus the $(0,1)$-form $\overline{\partial} f$ is "orthogonal" to a large collection of $(n, n-1)$-forms. Is this information enough to conclude that $\overline{\partial} f$ is the 0 form?

The proof involves three steps. First one shows that $\overline{\partial} f$ is orthogonal to a larger class of forms. Then one transfers the problem to the boundary. Finally a density lemma finishes the argument.

A preliminary observation is that the hypothesis of the theorem implies at least that $f$ is a (complex-valued) *harmonic* function in $\Omega$, for the coefficients of the Bochner–Martinelli kernel (1.15) are harmonic functions. The harmonicity of $f$ implies that the $(n-1, n)$ form $\overline{\partial} f(z) \wedge \sum_{j=1}^n d\overline{z}[j] \wedge dz[j]$ is a *closed* form in $\Omega$, and this property will be used in a moment.

By hypothesis $\overline{\partial} f$ has bounded coefficients in $\Omega$, so the explicit form of the Bochner–Martinelli kernel yields that the integral in (1.25) is a continuous function of $z$ not only for $z$ in $\Omega$ but for $z$ throughout $\mathbb{C}^n$.

**Exercise 1.3.11.** Verify the preceding statement. (The integral is a convergent improper integral, but none of the standard convergence theorems for integrals is directly applicable.)

In particular, the integral in (1.25) vanishes for $z$ on $b\Omega$. Since the integral represents a harmonic function of $z$ in the interior of the complement of $\Omega$, the integral vanishes on the bounded components of the complement of $\Omega$. Moreover, the explicit form of the Bochner–Martinelli kernel shows that the integral tends to 0 at infinity, so the integral also vanishes on the unbounded component of the complement of $\Omega$. In summary, the integral in (1.25) is identically equal to 0 for all $z$ in $\mathbb{C}^n$, not just for $z$ in $\Omega$.

In view of expression (1.17) for the Bochner–Martinelli kernel, an equivalent statement is that

$$\int_\Omega d\biggl(\frac{1}{|w-z|^{2n-2}}\,\overline{\partial} f(w) \wedge \sum_{j=1}^n d\overline{w}[j] \wedge dw[j]\biggr) = 0, \qquad z \in \mathbb{C}^n,$$

because, as observed above, the form $\overline{\partial} f(w) \wedge \sum_{j=1}^n d\overline{w}[j] \wedge dw[j]$ is closed. If $z \in \Omega$, then Stokes's theorem implies that

$$\lim_{\epsilon \to 0}\biggl(\int_{b\Omega} - \int_{bB(z,\epsilon)}\biggr)\frac{1}{|w-z|^{2n-2}}\,\overline{\partial} f(w) \wedge \sum_{j=1}^n d\overline{w}[j] \wedge dw[j] = 0.$$

Since the surface area of the boundary of the ball $B(z,\epsilon)$ is proportional to $\epsilon^{2n-1}$, while $|w-z|^{-(2n-2)} = O(\epsilon^{-(2n-2)})$ when $w \in bB(z,\epsilon)$, the integral over $bB(z,\epsilon)$ tends to 0 with $\epsilon$. Thus

$$\int_{b\Omega} \frac{1}{|w-z|^{2n-2}}\,\overline{\partial} f(w) \wedge \sum_{j=1}^{n} d\overline{w}[j] \wedge dw[j] = 0, \qquad z \in \Omega.$$

A similar argument, without taking a limit, shows that the preceding integral vanishes also when $z$ is outside the closure of $\Omega$. The following density lemma now comes into play.

**Lemma 1.3.12.** *Every continuous function $h(w)$ on $b\Omega$ can be uniformly approximated by linear combinations of the functions $1/|w-z|^{-(2n-2)}$, where $z \notin b\Omega$.*

Assume the lemma for the moment. The function $f$ is continuous on $b\Omega$ by hypothesis, so

$$\int_{b\Omega} \overline{f}(w)\,\overline{\partial} f(w) \wedge \sum_{j=1}^{n} d\overline{w}[j] \wedge dw[j] = 0.$$

The closedness of the form $\overline{\partial} f(w) \wedge \sum_{j=1}^{n} d\overline{w}[j] \wedge dw[j]$ implies by Stokes's theorem that

$$\int_{\Omega} \partial\overline{f}(w) \wedge \overline{\partial} f(w) \wedge \sum_{j=1}^{n} d\overline{w}[j] \wedge dw[j] = 0,$$

or, equivalently, that

$$\int_{\Omega} \sum_{j=1}^{n} \left| \frac{\partial f}{\partial \overline{z}_j} \right|^2 dV = 0.$$

Thus $f$ is holomorphic, as claimed.

To complete the proof of the theorem, it remains to prove Lemma 1.3.12. The idea of the argument goes back to a paper of M. V. Keldysh and M. A. Lavrent′ev [14] (reprinted in [13]). It suffices to approximate functions that are restrictions to $b\Omega$ of class $C^1$ functions in $\mathbb{C}^n$, because such functions are dense in the continuous functions on $b\Omega$ (by the Stone–Weierstrass theorem, for instance). Fix such a function $h$, and write the Bochner–Martinelli representation (1.18) for $h$ on a large ball $B$ that contains the closure of $\Omega$:

$$h(z) = \int_{bB} h(w)\,U(w,z) - \int_{B} \overline{\partial} h(w) \wedge U(w,z), \qquad z \in b\Omega.$$

To approximate $h(z)$ on $b\Omega$, approximate each of the integrals by Riemann sums. (The second integral is an absolutely convergent improper integral, so one can first delete from the integration region an arbitrarily small neighborhood of $b\Omega$, making an arbitrarily small error in the approximation.) In view of the expression (1.17) for the Bochner–Martinelli kernel, the terms in the Riemann sums are linear combinations of derivatives of the functions $1/|w-z|^{-(2n-2)}$ for $w \notin b\Omega$ (notice that the variables are interchanged from the statement of the lemma). Approximating those derivatives by difference quotients completes the proof of the Lemma 1.3.12. That concludes the proof of Theorem 1.3.10. $\square$

**Exercise 1.3.13.** Show that if $f$ is a class $C^1$ function on the closure of $\Omega$, where $\Omega$ has defining function $\rho$, then

$$\overline{\partial} f(z) \wedge \sum_{j=1}^{n} d\overline{z}[j] \wedge dz[j]\bigg|_{b\Omega} = \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial f}{\partial \overline{z}_j}\, d\overline{z} \wedge dz[j]\bigg|_{b\Omega}$$
$$= (-1)^{n(n+1)/2}(2i)^n |\nabla\rho|^{-1} \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}\frac{\partial \rho}{\partial z_j}\, d\mathsf{SurfaceArea}.$$

The expression $\displaystyle\sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}\frac{\partial \rho}{\partial z_j}$ is a "complex normal derivative" of $f$ on the boundary, and the proof of Theorem 1.3.10 may be interpreted as saying that this derivative vanishes identically on the boundary.

## 1.3.6 The spectrum for the ball

This section revisits the case of the unit ball $B$ in $\mathbb{C}^n$ to determine the eigenvalues and the eigenfunctions of the Bochner–Martinelli integral explicitly. First some background on spherical harmonics is needed.

A function $f$ is called *homogeneous of degree $k$* if $f(\lambda x) = \lambda^k f(x)$ when $\lambda > 0$ and $x$ is arbitrary. The restriction to the sphere (that is, the boundary of the ball) of a harmonic polynomial that is homogeneous of degree $k$ is called a *spherical harmonic of degree $k$*. In the plane, for example, the restriction to the unit circle of the function $xy$ is a spherical harmonic of degree 2. In $\mathbb{C}^2$, the restriction to the boundary of the unit ball of the function $z_1 z_2 \overline{z}_1^2 - z_2^2 \overline{z}_1 \overline{z}_2$ is a spherical harmonic of degree 4.

The notion of spherical harmonic makes sense in real space $\mathbb{R}^N$ as well as in complex space $\mathbb{C}^n$. In complex space, one can make the following finer gradation about the degree. If $s$ and $t$ are fixed nonnegative integers, and $q$ is a polynomial of the form $\sum_{|\alpha|=s}\sum_{|\beta|=t} c(\alpha,\beta) z^\alpha \overline{z}^\beta$, then one says that $q$ is homogeneous of bidegree $(s,t)$.

**Remark 1.3.14.** When $q$ is a polynomial in the underlying real coordinates, there is an ambiguity about how to denote the function $q$ as a function of the complex coordinates. One might write $q(z,\overline{z})$ to indicate that $q$ is a polynomial in $z$ and $\overline{z}$. Alternatively, since $z$ and $\overline{z}$ are not actually independent quantities, one might write $q(z)$ to mean the function $q$ evaluated at the point in $\mathbb{R}^{2n}$ corresponding to the point $z$ in $\mathbb{C}^n$. These notes use the second convention. One has to keep in mind, then, that $q(z)$ is not in general a holomorphic function of $z$ when $q$ is a "polynomial".

Let $H(s,t)$ denote the space of spherical harmonics of bidegree $(s,t)$, that is, the space of restrictions to the boundary of the unit ball of harmonic, homogeneous polynomials of bidegree $(s,t)$.

**Lemma 1.3.15.** *When $n \geq 2$, the Hilbert space $L^2(bB, d\mathsf{SurfaceArea})$ of square-integrable functions on the boundary of the unit ball is the orthogonal direct sum of the finite-dimensional subspaces $H(s,t)$ as $s$ and $t$ run over the non-negative integers.*

**Exercise 1.3.16.** Show that when $n = 1$, the space $H(s,t)$ is trivial when $s$ and $t$ are simultaneously nonzero, and the space of square-integrable functions on the unit circle is the orthogonal direct sum of the finite-dimensional subspaces $H(s,0)$ and $H(0,t)$ as $s$ and $t$ run over the non-negative integers. (The space $H(0,0)$ is counted only once.)

Assume the lemma for the moment. The next proposition says that the Bochner–Martinelli integral of a spherical harmonic of bidegree $(s,t)$ is very simple: the result is the same harmonic polynomial multiplied by a constant.

**Proposition 1.3.17.** *If $q$ is a harmonic polynomial of bidegree $(s,t)$, then the Bochner–Martinelli integral*

$$\int_{bB} q(w)\, U(w,z) = \frac{n+s-1}{n+s+t-1}\, q(z), \qquad z \in B. \tag{1.26}$$

*Proof.* The computation uses the following complex version of the Euler formula for homogeneous functions.

**Exercise 1.3.18.** If $q$ is a harmonic polynomial of bidegree $(s,t)$, then

$$\sum_{j=1}^{n} z_j \frac{\partial q}{\partial z_j} = sq(z) \qquad \text{and} \qquad \sum_{j=1}^{n} \overline{z}_j \frac{\partial q}{\partial \overline{z}_j} = tq(z).$$

*Hint*: Compute $\dfrac{d}{d\lambda}(\lambda z)^\alpha\, \overline{z}^\beta\big|_{\lambda=1}$ first by using the chain rule and then by using homogeneity.

The explicit formulas (1.23) and (1.24) for the Bochner–Martinelli kernel $U(w,z)$ and the Poisson kernel $P(w,z)$ of the ball show that

$$\begin{aligned}\int_{bB} q(w)\, U(w,z) &= \int_{bB} q(w) \frac{1-\langle w,z\rangle}{1-|z|^2}\, P(w,z) \\ &= \frac{1}{1-|z|^2}\left(q(z) - \sum_{j=1}^{n} \overline{z}_j \int_{bB} w_j q(w)\, P(w,z)\right), \qquad z \in B.\end{aligned} \tag{1.27}$$

The Poisson integral does not reproduce the polynomial $w_j q(w)$, because the harmonicity of $q(w)$ is not inherited by $w_j q(w)$. In fact,

$$\sum_{k=1}^{n} \frac{\partial^2}{\partial w_k \partial \overline{w}_k} (w_j q(w)) = \frac{\partial q}{\partial \overline{w}_j}.$$

The Poisson integral of $w_j q(w)$ should differ from $w_j q(w)$ by a function that has the same Laplacian but that vanishes at the boundary. Observe that

$$\sum_{k=1}^{n} \frac{\partial^2}{\partial w_k \partial \overline{w}_k} \left( (1 - |w|^2) \frac{\partial q}{\partial \overline{w}_j} \right) = -n \frac{\partial q}{\partial \overline{w}_j} - \sum_{k=1}^{n} \left( w_k \frac{\partial}{\partial w_k} + \overline{w}_k \frac{\partial}{\partial \overline{w}_k} \right) \frac{\partial q}{\partial \overline{w}_j}.$$

Since $\partial q / \partial \overline{w}_j$ has bi-homogeneity $(s, t-1)$, Exercise 1.3.18 implies that the right-hand side reduces to

$$-(n + s + t - 1) \frac{\partial q}{\partial \overline{w}_j}.$$

Consequently, the Poisson integral of $w_j q(w)$ equals the harmonic function

$$w_j q(w) + \frac{1}{n + s + t - 1} (1 - |w|^2) \frac{\partial q}{\partial \overline{w}_j}.$$

Thus (1.27) implies that

$$\int_{bB} q(w)\, U(w, z) = \frac{1}{1 - |z|^2} \left( q(z) - |z|^2 q(z) - \frac{1}{n + s + t - 1} (1 - |z|^2) \sum_{j=1}^{n} \overline{z}_j \frac{\partial q}{\partial \overline{z}_j} \right),$$

and another application of Exercise 1.3.18 shows that the right-hand side reduces to

$$q(z) \left( 1 - \frac{t}{n + s + t - 1} \right).$$

The desired conclusion (1.26) follows. □

Let $M$ denote the operator that takes a function on the boundary of the ball, computes the Bochner–Martinelli integral, and restricts the result to the boundary of the ball. It is not clear a priori that this operator $M$ makes sense on general square-integrable functions on the boundary. The preceding proposition, however, implies that $M$ is well defined on each subspace $H(s, t)$ as an operator of norm 1 or less. Lemma 1.3.15 implies that $M$ extends to $L^2(bB, d\mathsf{SurfaceArea})$ as a bounded operator of norm 1. Thus the Bochner–Martinelli integral of a square-integrable function on $bB$, which a priori exists only as a function inside $B$, has in a natural way boundary values in $L^2(bB, d\mathsf{SurfaceArea})$.

Exercise 1.3.16 implies that when $n = 1$, the eigenvalues of the operator $M$ are 0 and 1. When $n \geq 2$, the situation is completely different. The following theorem can be read off from the explicit form of the eigenvalues in Lemma 1.3.17 (with more or less ease, depending on what facts one knows from functional analysis).

**Theorem 1.3.19.** *Suppose $n \geq 2$. The Bochner–Martinelli operator $M$ acting on the space $L^2(bB, d\mathsf{SurfaceArea})$ has every rational number in the interval $(0, 1]$ as an eigenvalue of infinite multiplicity. The operator $M$ is a self-adjoint operator of norm 1. The spectrum of $M$ is the interval $[0, 1]$. The iterates $M^j$ ($j = 1, 2, \dots$) converge in the strong operator topology to the orthogonal projection onto the subspace of $L^2(bB, d\mathsf{SurfaceArea})$ consisting of boundary values of holomorphic functions.*

This theorem is due to A. V. Romanov [30]. The exposition above follows the book of Kytmanov [19, §5.1].

*Proof of Theorem 1.3.19.* If $r$ is a rational number in $(0, 1]$, then there are infinitely many ways to write $r$ in the form $p/q$, where $p$ and $q$ are integers greater than $n$ (not necessarily coprime). Proposition 1.3.17 implies that when $s = p + 1 - n$ and $t = q - p$, the rational number $p/q$ is an eigenvalue of $M$ on the subspace $H(s, t)$. Thus the rational number $r$ is an eigenvalue of infinite multiplicity.

Because of the orthogonal direct-sum decomposition of $L^2(bB, d\mathsf{SurfaceArea})$ given in Lemma 1.3.15, one can deduce properties of $M$ from the knowledge of the eigenvalues. Since all the eigenvalues are in the interval $(0, 1]$, and 1 is an eigenvalue, the operator $M$ has norm equal to 1. Since all the eigenvalues are real, $M$ is self-adjoint.

Since $M$ is self-adjoint, the spectrum of $M$ (that is, the set of complex numbers $\lambda$ for which $(M - \lambda I)$ is not invertible) is a closed subset of the real line. Hence the spectrum contains the closed interval $[0, 1]$. If $\lambda > 1$, then $\lambda$ is not in the spectrum of $M$, because $(M - \lambda I)^{-1}$ is represented by the convergent series $-\lambda^{-1} \sum_{j=0}^{\infty} (M/\lambda)^j$. When $\lambda < 0$, then $\lambda$ is not in the spectrum of $M$, because $(M - \lambda I)^{-1}$ is represented by the convergent series $-(\lambda - \frac{1}{2})^{-1} \sum_{j=0}^{\infty} (\lambda - \frac{1}{2})^{-j} (M - \frac{1}{2} I)^j$. Thus the spectrum of $M$ is precisely the interval $[0, 1]$.

What convergence of the iterates $M^j$ in the strong operator topology means is that for each fixed element $f$ of $L^2(bB, d\mathsf{SurfaceArea})$, the functions $M^j(f)$ converge in norm. In view of the orthogonal direct-sum decomposition of Lemma 1.3.15, and the property that the eigenvalues of $M$ are in the interval $(0, 1]$, it is evident that $M^j(f)$ converges to the projection of $f$ onto the eigenspace corresponding to eigenvalue 1. That eigenspace is the direct sum of the subspaces $H(s, 0)$, where $s \geq 0$: namely, the subspace of boundary values of holomorphic functions. This orthogonal projection from $L^2(bB, d\mathsf{SurfaceArea})$ onto the boundary values of holomorphic functions is the *Szegő projection* of the ball. $\square$

It remains to prove Lemma 1.3.15 about the decomposition of $L^2(bB, d\mathsf{SurfaceArea})$ into the direct sum of orthogonal subspaces of spherical harmonics of different bidegrees.

*Proof of Lemma 1.3.15.* To check the orthogonality, suppose that $q_1$ and $q_2$ are spherical harmonics of bidegrees $(s_1, t_1)$ and $(s_2, t_2)$. It suffices to show that

$$s_1 \int_{bB} q_1(w)\overline{q_2(w)}\, d\mathsf{SurfaceArea}_w = s_2 \int_{bB} q_1(w)\overline{q_2(w)}\, d\mathsf{SurfaceArea}_w$$

and

$$t_1 \int_{bB} q_1(w)\overline{q_2(w)}\, d\mathsf{SurfaceArea}_w = t_2 \int_{bB} q_1(w)\overline{q_2(w)}\, d\mathsf{SurfaceArea}_w.$$

The first case reduces to the second case via conjugation. To check the second case, observe by Exercise 1.3.18 that

$$t_1 \int_{bB} q_1(w)\overline{q_2(w)}\, d\mathsf{SurfaceArea}_w = \int_{bB} \sum_{j=1}^{n} \overline{w}_j \frac{\partial q_1}{\partial \overline{w}_j}\, \overline{q_2(w)}\, d\mathsf{SurfaceArea}_w.$$

Using $|w|^2 - 1$ as the defining function $\rho(w)$ for the unit ball, one sees by Exercise 1.3.13 that the integral on the right-hand side equals

$$(-1)^{n(n+1)/2} \frac{2}{(2i)^n} \int_{bB} \overline{q_2(w)}\, \overline{\partial} q_1(w) \wedge \sum_{j=1}^{n} d\overline{w}[j] \wedge dw[j].$$

Since $q_1$ is harmonic, the form $\overline{\partial} q_1(w) \wedge \sum_{j=1}^{n} d\overline{w}[j] \wedge dw[j]$ is closed, so Stokes's theorem converts the expression into

$$(-1)^{n(n+1)/2} \frac{2}{(2i)^n} \int_{B} \partial\overline{q_2(w)} \wedge \overline{\partial} q_1(w) \wedge \sum_{j=1}^{n} d\overline{w}[j] \wedge dw[j].$$

A parallel computation shows that $t_2 \int_{bB} q_1(w)\overline{q_2(w)}\, d\mathsf{SurfaceArea}_w$ equals the same expression multiplied by $(-1)^{n+1+(n-1)^2}$. Since that sign equals $+1$, the orthogonality is proved.

It remains to show that linear combinations of spherical harmonics are dense in the space of square-integrable functions on $bB$. Since polynomials are dense in the continuous functions on $bB$ (by the Stone–Weierstrass theorem), and hence in the square-integrable functions, it suffices to show that the restriction of an arbitrary homogeneous polynomial to the boundary of the ball is a linear combination of spherical harmonics.

**Lemma 1.3.20.** *If $q$ is a homogeneous polynomial in $\mathbb{C}^n$, then*

$$\Delta\left(|z|^{2j} q(z)\right) = |z|^{2j} \Delta q(z) + 4j(n + j - 1 + \deg(q))|z|^{2j-2} q(z).$$

*Proof.* Write the Laplace operator $\Delta$ in complex form and compute:

$$\sum_{l=1}^{n} 4\frac{\partial^2}{\partial z_l \partial \overline{z}_l}\left(\sum_{m=1}^{n} z_m \overline{z}_m\right)^j q(z) = \sum_{l=1}^{n} 4\frac{\partial}{\partial z_l}\left[|z|^{2j}\frac{\partial q}{\partial \overline{z}_l} + jq(z)|z|^{2j-2}z_l\right]$$
$$= \sum_{l=1}^{n} 4\left(|z|^{2j}\frac{\partial^2 q}{\partial z_l \partial \overline{z}_l} + j|z|^{2j-2}\overline{z}_l\frac{\partial q}{\partial \overline{z}_l} + j|z|^{2j-2}z_l\frac{\partial q}{\partial z_l} + jq(z)|z|^{2j-2} + j(j-1)q(z)|z|^{2j-4}z_l\overline{z}_l\right).$$

The first term of the sum equals $|z|^{2j}\Delta q(z)$. By Exercise 1.3.18, the next two terms of the sum simplify to $4j\deg(q)q(z)|z|^{2j-2}$. The fourth term yields $4jnq(z)|z|^{2j-2}$, and the final term contributes $4j(j-1)q(z)|z|^{2j-2}$. The equation in the lemma follows. □

With the lemma in hand, one can obtain the following representation for homogeneous polynomials.

**Exercise 1.3.21.** There are numbers $b_j(k)$ such that for every homogeneous polynomial $p$ in $\mathbb{C}^n$ of degree $k$, the difference

$$p(z) - \sum_{j=1}^{\lfloor k/2 \rfloor} b_j(k)|z|^{2j}\Delta^j p(z)$$

is harmonic. The numbers $b_j(k)$ may be determined recursively: $1/b_1(k) = 4(n+k-2)$, and $-b_j/b_{j+1} = 4(j+1)(n+k-j-2)$ when $j \geq 1$.

The exercise implies that the restriction of a homogeneous polynomial of degree $k$ to the boundary of the ball is equal to a spherical harmonic of degree $k$ plus a sum of restrictions of polynomials of lower degrees (since $|z|^2 = 1$ on the boundary). It follows by induction that the restriction of a homogeneous polynomial of arbitrary degree to the boundary of the ball is a linear combination of spherical harmonics (the base cases of degrees 0 and 1 being trivial). □

Exercise 1.3.21 is a naive, computational way to see that every homogeneous polynomial equals a harmonic homogeneous polynomial plus $|z|^2$ times a polynomial of lower degree. For a slick—but less intuitive—way to see this property, consult the book of Stein and Weiss [36, Chapter IV, §2] or the book of Rudin [32, §12.1].

### Further results and questions

When $\Omega$ is a general bounded domain, it is a priori conceivable either that the iterates $M^j$ on $L^2(b\Omega, d\mathsf{SurfaceArea})$ fail to converge or that the iterates converge to a non-orthogonal projection. As far as I know, this general situation remains to be worked out.

**Open problem 1.3.22.** If the iterates $M^j$ of the Bochner–Martinelli operator converge in the strong operator topology to the Szegő projection in $L^2(b\Omega, d\mathsf{SurfaceArea})$, is $\Omega$ necessarily a ball?

A uniqueness theorem of E. Ligocka [25] suggests to me that the answer to the preceding question should be affirmative.

One can also consider iterates of the Bochner–Martinelli operator in other function spaces. In his Seoul lecture notes [20, p. 20], Kytmanov posed the following problem, which as far as I know is still open.

**Open problem 1.3.23.** When $1 < p < \infty$, do the iterates $M^j$ of the Bochner–Martinelli operator for the ball converge in the strong operator topology to the Szegő projection in $L^p(b\Omega, d\mathsf{SurfaceArea})$?

One can also consider the Bochner–Martinelli operator $M$ as acting on functions on the interior: namely, restrict the function to the boundary (assuming that this restriction makes sense) and integrate against the Bochner–Martinelli kernel on the boundary to get a new function on the interior. When $k$ is a positive integer, let $W^k(B)$ denote the *Sobolev space* consisting of functions whose derivatives through order $k$ are square-integrable on $B$. The inner product in $W^k(B)$ can be taken to be the sum of the inner products in $L^2(B)$ of the derivatives.

**Exercise 1.3.24.** Prove that harmonic homogeneous polynomials of different bidegrees are orthogonal in $W^k(B)$.

It can be shown that functions in $W^k(B)$ have boundary values ("traces") in $L^2(bB)$ (actually the traces belong to the subspace $W^{k-\frac{1}{2}}(bB)$), and the map that takes a function $f$ to the Poisson integral of its boundary values is continuous in $W^k(B)$. Consequently, to study the Bochner–Martinelli operator in $W^k(B)$, it suffices to consider the action of $M$ on harmonic functions. Proposition 1.3.17 implies that the iterates $M^j$ converge in the strong operator topology of $W^k(B)$ to a projection operator onto the subspace of $W^k(B)$ consisting of holomorphic functions.

**Open problem 1.3.25.** For which domains $\Omega$ (other than balls) is it true that for every positive integer $k$ the iterates of the Bochner–Martinelli operator converge in $W^k(\Omega)$ to a projection operator onto the holomorphic subspace of $W^k(\Omega)$?

A. V. Romanov [31] obtained the following partial result.

**Theorem 1.3.26.** *Suppose $n \geq 2$. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with connected boundary of class $C^\infty$. The iterates of the Bochner–Martinelli operator converge in the strong operator topology of $W^1(\Omega)$ to a projection onto the subspace of $W^1(\Omega)$ consisting of holomorphic functions.*

Emil J. Straube observed that there is a necessary condition for a domain to satisfy the property indicated in Problem 1.3.25, and there are known domains that fail to satisfy that necessary condition. Namely, the property indicated in Problem 1.3.25 implies that there is a projection operator onto holomorphic functions that is continuous in $W^k(\Omega)$ for every positive integer $k$. It follows that the holomorphic subspace of $W^k(\Omega)$ is dense in the holomorphic subspace of $W^1(\Omega)$ for every positive integer $k$. Barrett and Fornæss [4] have given an example of a (non-pseudoconvex) Hartogs domain in $\mathbb{C}^2$ with smooth boundary for which (in particular) the holomorphic subspace of $W^5(\Omega)$ is not dense in the holomorphic subspace of $W^1(\Omega)$, even in the topology of uniform convergence on compact subsets of $\Omega$.

### 1.3.7 CR functions and extension from the boundary

Section 1.3.4 shows that if $\Omega$ is a bounded domain in $\mathbb{C}^n$, where $n \geq 2$, and if the boundary of $\Omega$ is connected, then every function holomorphic in a connected neighborhood of the boundary $b\Omega$ extends holomorphically to all of $\Omega$. It is natural to ask if an extension phenomenon still exists in the limit as the thickness of the neighborhood of the boundary goes to 0.

There is a technical problem in trying to formulate such a theorem, since holomorphic functions live on open sets. One needs a property of functions living on the boundary $b\Omega$ that characterizes boundary values of holomorphic functions.

In complex space $\mathbb{C}$ of dimension 1, there is no local differential condition that describes a linear space of restrictions of holomorphic functions to a curve. Indeed, consider the following observations about classes of functions defined on an open segment $S$ of the real axis. (a) Every real-analytic function on $S$ is the restriction to $S$ of a function holomorphic in an open neighborhood of $S$. (b) By the Schwarz reflection principle, a *real-valued* continuous function on $S$ is the continuous boundary value of a holomorphic function on one side of $S$ if and only if the boundary function is real-analytic. (c) Every complex-valued, Hölder-continuous function on $S$ can be expressed as the difference of boundary values on $S$ of a holomorphic function in the upper half-plane and a holomorphic function in the lower half-plane. The latter fact follows from the Plemelj jump formula for the Cauchy integral (also known as the Sokhotskiĭ formula), for which see [26, vol. I, §74], for example.

In complex space of higher dimension, however, the trace of a holomorphic function on a hypersurface must satisfy the Cauchy–Riemann equations in tangential complex directions. That observation motivates the definition of CR functions.

**Theorem/Definition 1.3.27.** Suppose $n \geq 2$. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with class $C^1$ boundary. A continuous function $f$ on the boundary $b\Omega$ is called a *CR function* on $b\Omega$ if the following property holds.

1. For every differential form $\varphi$ of type $(n, n-2)$ and class $C^1(\mathbb{C}^n)$, the integral

$$\int_{b\Omega} f\,\overline{\partial}\varphi = 0. \tag{1.28}$$

If $f$ is a class $C^1$ function on $b\Omega$ (in other words, there is a function $F$ of class $C^1(\mathbb{C}^n)$ such that $F\big|_{b\Omega} = f$), and if $\rho$ is a class $C^1$ defining function for $\Omega$, then each of the following properties is equivalent to property 1.

2. For every extension $F$ of $f$, and for all integers $j$ and $k$ between 1 and $n$, the expression

$$\frac{\partial F}{\partial \overline{z}_j}\frac{\partial \rho}{\partial \overline{z}_k} - \frac{\partial F}{\partial \overline{z}_k}\frac{\partial \rho}{\partial \overline{z}_j}$$

   is equal to 0 at every point of $b\Omega$.

3. For every extension $F$ of $f$, the differential form $\overline{\partial}F \wedge \overline{\partial}\rho$ is equal to 0 at every point of $b\Omega$.

4. For every extension $F$ of $f$, for every point $w$ in $b\Omega$, and for all complex numbers $t_1$, …, $t_n$ such that $\sum_{j=1}^n t_j(\partial\rho/\partial\overline{z}_j)(w) = 0$, the expression

$$\sum_{j=1}^n t_j \frac{\partial F}{\partial \overline{z}_j}(w) \qquad \text{is equal to 0.}$$

The properties in the preceding definition are local (in equation (1.28), one can restrict attention to forms $\varphi$ with support in a small open set), so there is a corresponding concept of a function being a CR function on an open portion of $b\Omega$.

*Proof of the equivalence.* For every extension $F$ of $f$, degree considerations imply that

$$0 = \int_{b\Omega} d(F\,\varphi) = \int_{b\Omega} \overline{\partial}(F\,\varphi) = \int_{b\Omega} \overline{\partial}F \wedge \varphi + \int_{b\Omega} F\,\overline{\partial}\varphi.$$

Therefore property 1 is equivalent to the statement that $\int_{b\Omega} \overline{\partial}F \wedge \varphi = 0$. Fix indices $j$ and $k$, assume without loss of generality that $j < k$, and choose $\varphi$ of the form $\psi\, d\overline{z}[j,k] \wedge dz$, where $\psi$ is an arbitrary function of class $C^1(\mathbb{C}^n)$. Then

$$\overline{\partial}F \wedge \varphi = \psi(z)\left((-1)^{j-1}\frac{\partial F}{\partial \overline{z}_j}\, d\overline{z}[k] \wedge dz + (-1)^{k-2}\frac{\partial F}{\partial \overline{z}_k}\, d\overline{z}[j] \wedge dz\right).$$

In view of Exercise 1.2.2, property 1 implies that

$$\int_{b\Omega} \psi(z)(-1)^{j+k}|\nabla\rho(z)|^{-1}\left(\frac{\partial F}{\partial\overline{z}_j}\frac{\partial\rho}{\partial\overline{z}_k} - \frac{\partial F}{\partial\overline{z}_k}\frac{\partial\rho}{\partial\overline{z}_j}\right) d\mathsf{SurfaceArea} = 0.$$

Since $\psi$ is arbitrary, it follows that property 1 implies property 2. On the other hand, an arbitrary $\varphi$ is a sum of differential forms of the kind just considered, so the same argument shows that property 2 implies property 1.

The equivalence of properties 2 and 3 is a direct calculation:

$$\overline{\partial}F \wedge \overline{\partial}\rho = \sum_{j<k}\left(\frac{\partial F}{\partial\overline{z}_j}\frac{\partial\rho}{\partial\overline{z}_k} - \frac{\partial F}{\partial\overline{z}_k}\frac{\partial\rho}{\partial\overline{z}_j}\right) d\overline{z}_j \wedge d\overline{z}_k.$$

The definition of class $C^1$ boundary entails that $\nabla\rho \neq 0$ on $b\Omega$. Consequently, for each point $w$ in $b\Omega$ there is some index $k$ for which $(\partial\rho/\partial\overline{z}_k)(w) \neq 0$. Then the expressions

$$\frac{\partial\rho}{\partial\overline{z}_k}(w)\frac{\partial}{\partial\overline{z}_j} - \frac{\partial\rho}{\partial\overline{z}_j}(w)\frac{\partial}{\partial\overline{z}_k} \qquad (\text{where } j \in \{1,\dots,k-1,k+1,\dots,n\}),$$

which represent tangent vectors at $w$, form a basis for the $(n-1)$-dimensional space of vectors $\sum_{j=1}^{n} t_j(\partial/\partial\overline{z}_j)$ with the property that $\sum_{j=1}^{n} t_j(\partial\rho/\partial\overline{z}_j)(w) = 0$. Therefore properties 2 and 4 are equivalent. $\square$

**Remark 1.3.28.** The proof reveals that in properties 2–4 of Theorem/Definition 1.3.27, one can equivalently replace the words "for every extension" by the words "for one extension". Moreover, the proof is unchanged if the condition that $F$ is in class $C^1(\mathbb{C}^n)$ is replaced by the following weaker condition:

$F$ is continuous on $\mathbb{C}^n$, the restriction $F\big|_{b\Omega}$ equals $f$, the first partial derivatives of $F$ exist on $b\Omega$, and the first partial derivatives of $F$ are continuous on $b\Omega$. (1.29)

That weaker condition on the extension will come into play in a moment.

Theorem/Definition 1.3.27 characterizes CR-functions by saying that for an arbitrary extension $F$, the differential form $\overline{\partial}F$ has only a "complex normal component" at the boundary. By choosing a special extension $F$, one can eliminate this normal component.

**Proposition 1.3.29.** *Suppose $n \geq 2$. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with class $C^1$ boundary. A class $C^1$ function $f$ on $b\Omega$ is a CR function if and only if there exists an extension $F$ satisfying* (1.29) *such that $\overline{\partial}F = 0$ at each point of $b\Omega$.*

*Proof.* If such an extension $F$ exists, then $f$ is a CR function by property 3 of Theorem/Definition 1.3.27 (in view of Remark 1.3.28). The new content of the proposition is the converse direction.

Let $\rho$ be a class $C^1$ defining function for $\Omega$, and let $F_1$ be an arbitrary extension of $f$ to a function of class $C^1(\mathbb{C}^n)$. The idea is to correct $F_1$ by subtracting a suitable term that vanishes at the boundary $b\Omega$ and that cancels out the component of $\overline{\partial}\rho$ in $\overline{\partial}F_1$. It suffices to prescribe $F$ in a neighborhood of $b\Omega$ where $\nabla\rho \neq 0$. Define $F$ in such a neighborhood as follows:

$$F(z) = F_1(z) - \frac{4\rho(z)}{|\nabla\rho(z)|^2}\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}\frac{\partial F_1}{\partial \overline{z}_j}. \tag{1.30}$$

Evidently $F\big|_{b\Omega} = F_1\big|_{b\Omega}$. Moreover, the $d\overline{z}_k$ component of $\overline{\partial}F$ at a point of $b\Omega$ equals

$$\frac{\partial F_1}{\partial\overline{z}_k} - \left(\frac{4}{|\nabla\rho|^2}\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}\frac{\partial F_1}{\partial\overline{z}_j}\right)\frac{\partial\rho}{\partial\overline{z}_k}.$$

By property 2 of Theorem/Definition 1.3.27, the preceding expression equals

$$\frac{\partial F_1}{\partial\overline{z}_k} - \left(\frac{4}{|\nabla\rho|^2}\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_j}\right)\frac{\partial F_1}{\partial\overline{z}_k},$$

which reduces to 0. Thus $F$ has the required properties. □

**Exercise 1.3.30** (suggested by Tao Mei)**.** Show that when $n = 1$, the conclusion of Proposition 1.3.29 holds for *every* function $f$ of class $C^1$ on $b\Omega$ (the condition that $f$ is a CR function being vacuous when $n = 1$).

This preparation leads to the following boundary version of the extension phenomenon of Hartogs. For several decades, starting around 1965, the result was known in the literature as "Bochner's theorem" or as the "Hartogs–Bochner extension theorem", but Range has documented that this attribution is mistaken [28].

**Theorem 1.3.31.** *Suppose $n \geq 2$. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with connected boundary of class $C^1$. If $f$ is a continuous CR function on $b\Omega$, then there exists a (unique) function $F$, continuous on the closure of $\Omega$, such that $F$ is holomorphic in $\Omega$ and $F\big|_{b\Omega} = f$. If $f$ is class $C^1$ on $b\Omega$, then $F$ is class $C^1$ on the closure of $\Omega$.*

*Proof.* It is easy to see that $F$ is unique, for the difference of two such functions is a holomorphic function in $\Omega$ that is identically equal to 0 on the boundary. By the maximum principle, the difference is identically equal to 0 in $\Omega$.

As in the proof of Theorem 1.3.7, there is a natural way to define a candidate for $F$ via the Bochner–Martinelli integral: namely

$$F(z) := \int_{b\Omega} f(w)\, U(w,z), \qquad z \notin b\Omega.$$

Three properties need to be verified: first, that $F$ is holomorphic in $\Omega$; second, that $F$ is a continuous extension of $f$; and third, that $F$ is class $C^1$ on the closure of $\Omega$ if $f$ is class $C^1$ on $b\Omega$.

According to Lemma 1.3.8, there is a differential form $V_k(w,z)$ of type $(n, n-2)$ in $w$, smooth for $w \neq z$, such that

$$\frac{\partial F}{\partial \overline{z}_k} = \int_{b\Omega} f(w)\, \overline{\partial}_w V_k(w,z).$$

Since $f$ is a CR function, the integral on the right-hand side is equal to 0 by property 1 of Theorem/Definition 1.3.27. Thus $F$ is holomorphic both inside $\Omega$ and outside the closure of $\Omega$. Since $n \geq 2$, the same argument as in section 1.3.3 shows that $F$ is identically equal to 0 on the complement of the closure of $\Omega$. (This step uses that $b\Omega$ is connected and class $C^1$, so the complement of the closure of $\Omega$ is connected.)

In particular, $F$ has a limit on $b\Omega$ from the outside, and this limit is identically equal to 0. The main part of the proof consists in showing that $F$ has a limit at $b\Omega$ from the inside and that this limit is equal to $f$. Since $f$ is uniformly continuous on $b\Omega$, it suffices to prove that a limit from the inside is taken along each normal line, uniformly along the boundary.

Fix a positive $\epsilon$. The goal is to specify a small positive number $\delta$ and a positive number $C$ such that if $z$ is any point in $b\Omega$, and $\nu$ is the inner unit normal to the boundary at $z$, then $|F(z+t\nu) - f(z)| < C\epsilon$ when $0 < t < \delta$. (To start with, one should choose $\delta$ at least small enough to guarantee that the point $z + t\nu$ does lie inside $\Omega$ when $0 < t < \delta$.) The Bochner–Martinelli integral reproduces constant functions, so

$$F(z+t\nu) - f(z) = \int_{b\Omega} (f(w) - f(z))\, U(w, z+t\nu).$$

As observed above, the Bochner–Martinelli integral of a CR function is identically equal to 0 outside the closure of $\Omega$, so for small positive $t$ the preceding equation is equivalent to the following equation:

$$F(z+t\nu) - f(z) = \int_{b\Omega} (f(w) - f(z))\, (U(w, z+t\nu) - U(w, z-t\nu)).$$

Since $f$ is uniformly continuous on $b\Omega$, there is a positive number $\gamma$, independent of $z$ and $w$, such that $|f(w) - f(z)| < \epsilon$ when $w$ is in $b\Omega \cap B(z,\gamma)$. Additionally choose $\gamma$ small enough that the part of $b\Omega$ in $B(z,\gamma)$ is a graph over the tangent plane at $z$, and the scalar product of $(w-z)$ with $\nu$ has magnitude no greater than $\frac{1}{2}|w-z|$ for $w$ in $B(z,\gamma)$. Split the preceding integral into the part over $b\Omega \setminus B(z,\gamma)$ and the part over $b\Omega \cap B(z,\gamma)$. The first integral evidently tends to 0 with $t$ (uniformly with respect to $z$) because the integral does not see the singularity of the Bochner–Martinelli kernel. Indeed, that integral has absolute value bounded by $t$ times a quantity that depends only on $\gamma$, the dimension $n$, the maximum of $|f|$, and the total surface area of $b\Omega$. Having fixed $\gamma$, choose $\delta$ such that this first integral is less than $\epsilon$ when $0 < t < \delta$. The second integral has absolute value bounded by a constant times $\epsilon$ times

$$\sum_{j=1}^{n} \int_{b\Omega\cap B(z,\gamma)} \left| \frac{(w-z-t\nu)_j}{|w-z-t\nu|^{2n}} - \frac{(w-z+t\nu)_j}{|w-z+t\nu|^{2n}} \right| d\mathsf{SurfaceArea}. \tag{1.31}$$

To complete the proof, it suffices to show that the terms in the preceding sum are bounded independently of $z$ and $t$.

The choice of $\gamma$ implies that $|w-z\pm t\nu|^2 \ge \frac{1}{2}(|w-z|^2+t^2)$ when $w\in b\Omega\cap B(z,\gamma)$. Therefore

$$\left| \frac{(-t\nu)_j}{|w-z-t\nu|^{2n}} - \frac{(t\nu)_j}{|w-z+t\nu|^{2n}} \right| \le 2^{n+1} \frac{t}{(|w-z|^2+t^2)^n}.$$

Moreover, $\big|\,|w-z-t\nu|^2 - |w-z+t\nu|^2\,\big| \le 4t|w-z|$ for all $w$ and $z$, so

$$\left| \frac{1}{|w-z-t\nu|^{2}} - \frac{1}{|w-z+t\nu|^{2}} \right| \le \frac{16t|w-z|}{(|w-z|^2+t^2)^2}.$$

Since $|A^n - B^n| \le n|A-B|\max(|A|^{n-1},|B|^{n-1})$, it follows that

$$\left| \frac{(w-z)_j}{|w-z-t\nu|^{2n}} - \frac{(w-z)_j}{|w-z+t\nu|^{2n}} \right| \le \frac{16n2^{n-1}t|w-z|^2}{(|w-z|^2+t^2)^{n+1}} \le 2^{n+3}n\frac{t}{(|w-z|^2+t^2)^n}.$$

Consequently, each integrand in (1.31) is bounded by a constant (depending only on the dimension $n$) times $t/(|w-z|^2+t^2)^n$. Parametrizing the integral by using the coordinates in the tangent plane as parameters shows that the integral is bounded (independently of $z$) by a constant times

$$\int_{|x|<\gamma} \frac{t}{(|x|^2+t^2)^n}\, d\mathsf{Volume}_x,$$

where the parameter $x$ lies in $\mathbb{R}^{2n-1}$. Replacing the variable $x$ by $tu$ shows that the latter integral is bounded above by

$$\int_{\mathbb{R}^{2n-1}} \frac{1}{(|u|^2+1)^n}\, d\mathsf{Volume}_u,$$

independently of both $t$ and $\gamma$. Integrating in spherical coordinates shows that this integral converges. This estimate completes the proof that $F$ is continuous on the closure of $\Omega$, and $F\big|_{b\Omega} = f$.

It remains to show that $F$ is class $C^1$ on the closure of $\Omega$ when $f$ is class $C^1$ on $b\Omega$. The key point is that each derivative $\partial/\partial z_j$ essentially commutes with the Bochner–Martinelli integral of a CR function.

Proposition 1.3.29 provides a special extension $g$ of $f$ with the property that $\overline{\partial} g = 0$ on $b\Omega$. Now

$$\frac{\partial F}{\partial z_j} = \int_{b\Omega} g(w)\,\frac{\partial}{\partial z_j} U(w,z) = -\int_{b\Omega} g(w)\,\frac{\partial}{\partial w_j} U(w,z), \tag{1.32}$$

since the Bochner–Martinelli kernel depends only on the difference $w - z$. Introduce the ad hoc notation $U_j(w,z)$ for the $(n-1, n-1)$ form such that $U(w,z) = dw_j \wedge U_j(w,z)$. Then $\frac{\partial}{\partial w_j} U(w,z) = \partial_w U_j(w,z)$. Since $U(w,z)$ is $\overline{\partial}_w$-closed, so is $U_j(w,z)$, and therefore $\frac{\partial}{\partial w_j} U(w,z) = d_w U_j(w,z)$. Now rewrite equation (1.32) as follows:

$$\frac{\partial F}{\partial z_j} = -\int_{b\Omega} g(w)\, d_w U_j(w,z) = \int_{b\Omega} dg(w) \wedge U_j(w,z).$$

Since $\overline{\partial} g = 0$ on $b\Omega$, one has that

$$\frac{\partial F}{\partial z_j} = \int_{b\Omega} \partial g(w) \wedge U_j(w,z) = \int_{b\Omega} \frac{\partial g}{\partial w_j}\, dw_j \wedge U_j(w,z) = \int_{b\Omega} \frac{\partial g}{\partial w_j}\, U(w,z).$$

Thus the function $\partial F/\partial z_j$ is the Bochner–Martinelli integral of the continuous CR function $\partial g/\partial w_j$.

**Exercise 1.3.32.** Explain why $\partial g/\partial w_j$ is a CR function on $b\Omega$.

By what was shown above, the function $\partial F/\partial z_j$ is a continuous extension of $\partial g/\partial w_j$ to $\Omega$. Moreover, the derivative $\partial F/\partial \overline{z}_j$ is identically equal to 0, since $F$ is holomorphic in $\Omega$. Since the index $j$ is arbitrary, every first-order partial derivative of $F$ has a continuous extension to the closure of $\Omega$. Thus the function $F$ is class $C^1$ on the closure of $\Omega$. $\square$

**Exercise 1.3.33.** Show that if in Theorem 1.3.31 the boundary of $\Omega$ is class $C^k$ (where $k \geq 2$) and the function $f$ is class $C^k$ on $b\Omega$, then the extension $F$ is class $C^k$ on the closure of $\Omega$.

The hypothesis in Theorem 1.3.31 of connectedness of the boundary is essential. Consider, for example, a domain bounded by two concentric spheres. A function that is equal to different constants on the two boundary components is a CR function that does not extend to a holomorphic function on the domain.

By strengthening the hypotheses, however, one can obtain a version of Theorem 1.3.31 that does apply to domains with (possibly) disconnected boundary. The following result is due to Barnet M. Weinstock [37] (under more restrictive differentiability hypotheses).

**Exercise 1.3.34.** Suppose $n \geq 2$. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with (not necessarily connected) boundary of class $C^1$. If $f$ is a continuous function on $b\Omega$ with the property that

$$\int_{b\Omega} f\,\omega = 0$$

for every differential form $\omega$ of type $(n, n-1)$ with coefficients in class $C^1$ on the closure of $\Omega$ such that $\overline{\partial}\omega = 0$ on $\Omega$, then there exists a (unique) function $F$, continuous on the closure of $\Omega$, such that $F$ is holomorphic in $\Omega$ and $F\big|_{b\Omega} = f$.

Another situation to which the statement of Theorem 1.3.31 fails to apply is the case when the dimension $n$ is equal to 1 (for then the concept of CR function is vacuous). The following exercise gives a substitute for the theorem when $n = 1$.

**Exercise 1.3.35.** Prove that if $\Omega$ is a bounded domain in the plane $\mathbb{C}$ with connected boundary $b\Omega$ of class $C^1$, and if $f$ is a continuous function on $b\Omega$ such that

$$\int_{b\Omega} f(z) z^k \, dz = 0 \qquad \text{when } k \geq 0,$$

then there exists a (unique) function $F$, continuous on the closure of $\Omega$, such that $F$ is holomorphic in $\Omega$ and $F\big|_{b\Omega} = f$. If $f$ is class $C^1$ on $b\Omega$, then $F$ is class $C^1$ on the closure of $\Omega$.

## Further results

The preceding results about extension of CR functions assume that the domain is bounded. The one place where that assumption is used in a crucial way is to guarantee that when $|z| \to \infty$, the Bochner–Martinelli kernel $U(w, z)$ tends to 0 uniformly with respect to $w$ in $b\Omega$. That deduction breaks down if $b\Omega$ is unbounded, and the theorems break down too.

**Example 1.3.36.** Let $\Gamma$ denote the flat hypersurface on which $\operatorname{Re} z_1 = 0$. Every continuous function $f$ depending only on $\operatorname{Im} z_1$ is a CR function on $\Gamma$. If $f$ is the continuous boundary value of a function that is holomorphic on one side of $\Gamma$, then $f$ has to be real-analytic because of the Schwarz reflection principle. But there exist functions of class $C^\infty$ that are not real analytic. Consequently, CR functions on a flat hypersurface do not necessarily extend holomorphically to either side of the hypersurface.

Nonetheless, there are cases in which CR functions do extend from the boundary of an unbounded domain into the domain. The model example is the so-called Siegel upper half-space in $\mathbb{C}^n$ consisting of points $(z_1, \dots, z_n)$ such that $\operatorname{Im} z_1 > |z_2|^2 + \cdots + |z_n|^2$. This unbounded domain is biholomorphically equivalent to the unit ball (via the Cayley transform), and CR functions on the boundary of the Siegel upper half-space do extend to be holomorphic functions inside the domain.

Evidently one needs some sort of (pseudo)convexity hypothesis in the case of an unbounded domain. The following theorem is a version of a result of M. Naser Šafii [33]. The differentiability hypotheses in the statement can be weakened, but a little extra differentiability reduces the technical prerequisites for the proof. (The original statement and proof are phrased in the language of currents.)

**Theorem 1.3.37.** *Suppose $n \geq 2$. Let $\Omega$ be a domain in $\mathbb{C}^n$, possibly unbounded, with connected boundary of class $C^4$. Suppose additionally that the envelope of holomorphy of $\mathbb{C}^n \setminus$ closure$(\Omega)$ equals $\mathbb{C}^n$; in other words, every holomorphic function on the complement of the closure of $\Omega$ extends holomorphically to $\mathbb{C}^n$. Then for every CR function $f$ on $b\Omega$ of class $C^4$, there exists a (unique) function $F$, continuous on the closure of $\Omega$, such that $F$ is holomorphic in $\Omega$ and $F\big|_{b\Omega} = f$.*

*Proof.* The idea in the proof is similar to the strategy in the proof of Theorem 1.3.7 about extending holomorphic functions across compact holes. First extend the function $f$ arbitrarily, and then solve a $\overline{\partial}$-problem to correct the extension to ensure holomorphicity. The main complication is that in the new setting, the theorem about solving the $\overline{\partial}$-problem with compact support is not directly applicable. To handle this complication, one needs the extra hypothesis about the envelope of holomorphy together with the following standard result (whose proof is deferred until a later section).

**Lemma 1.3.38.** *If $G$ is a $\overline{\partial}$-closed $(0,1)$-form of class $C^1$ on the whole space $\mathbb{C}^n$, then there exists a function $u$ of class $C^1$ on $\mathbb{C}^n$ such that $\overline{\partial} u = G$.*

Also needed in the proof is the following refinement of Proposition 1.3.29. It is here that the extra differentiability is convenient.

**Lemma 1.3.39.** *Let $\Omega$ be a domain in $\mathbb{C}^n$, possibly unbounded, with boundary defining function $\rho$ of class $C^4$. If $f$ is a CR function on $b\Omega$ of class $C^4$, then there exist a function $F$ on $\mathbb{C}^n$ of class $C^2$ such that $F\big|_{b\Omega} = f$ and a $(0,1)$-form $h$ on $\mathbb{C}^n$ of class $C^1$ such that $\overline{\partial} F = \rho^2 h$.*

*Proof.* Although stated globally, the lemma is essentially local. If one can find the required functions $F$ and $h$ in a neighborhood of a point of $b\Omega$, then one can patch with a partition of unity to obtain global functions.

The proof of Proposition 1.3.29 (in particular, equation (1.30)) provides a function $g_1$ of class $C^3$ such that $g_1\big|_{b\Omega} = f$ and $\overline{\partial} g_1\big|_{b\Omega} = 0$. What is needed now is a further correction to make $\overline{\partial}$ of the extension vanish at $b\Omega$ to order $\rho^2$.

From (1.30), one sees that there is a $(0,1)$-form $\theta_1$ of class $C^2$ such that $\overline{\partial} g_1 = \rho\theta_1$. Since $\overline{\partial}(\overline{\partial} g_1) = 0$, one has that $\theta_1 \wedge \overline{\partial}\rho = 0$ on $b\Omega$. Therefore there are a function $g_2$ of class $C^2$ and a $(0,1)$-form $\theta_2$ of class $C^1$ such that $\theta_1 = g_2\,\overline{\partial}\rho + \rho\theta_2$. If $F := g_1 - \frac{1}{2} g_2 \rho^2$, then $F$ is an extension of $f$ of class $C^2$ such that

$$\overline{\partial} F = \overline{\partial} g_1 - g_2 \rho\,\overline{\partial}\rho - \tfrac{1}{2}\rho^2\,\overline{\partial} g_2 = \rho^2(\theta_2 - \tfrac{1}{2}\,\overline{\partial} g_2).$$

Set $h$ equal to $\theta_2 - \frac{1}{2}\,\overline{\partial} g_2$ to obtain the statement of the lemma. □

**Exercise 1.3.40.** Suppose that $b\Omega$ and $f$ have a higher degree of differentiability. Can you construct an extension $F$ of $f$ such that $\overline{\partial} F = O(\rho^3)$?

The function $F$ constructed in the proof of Lemma 1.3.39 is not the $F$ required in the conclusion of the theorem, so rename the function coming from the lemma as $\phi$. The desired $F$ will be a further correction of $\phi$. Define a $(0,1)$-form $G$ in $\mathbb{C}^n$ by setting $G$ equal to $\overline{\partial}\phi$ in $\Omega$ and 0 in the complement of $\Omega$. Since $\overline{\partial}\phi = O(\rho^2)$, the $(0,1)$-form $G$ is a $\overline{\partial}$-closed form in $\mathbb{C}^n$ of class $C^1$.

According to Lemma 1.3.38, there is a function $u$ of class $C^1$ on $\mathbb{C}^n$ such that $\overline{\partial} u = G$. Since $G = 0$ in the complement of $\Omega$, the function $u$ is a holomorphic function on the complement of the closure of $\Omega$. The hypothesis about the envelope of holomorphy implies that there is a holomorphic function $v$ in all of $\mathbb{C}^n$ such that $v$ and $u$ are equal on the complement of the closure of $\Omega$. Since both $u$ and $v$ are continuous on $\mathbb{C}^n$, the two functions agree on $b\Omega$.

Define $F$ in $\Omega$ such that $F = \phi - u + v$. By construction, $\overline{\partial} F = \overline{\partial}\phi - \overline{\partial} u = 0$ in $\Omega$, so $F$ is a holomorphic function in $\Omega$. Moreover, $(-u+v)\big|_{b\Omega} = 0$, so $F\big|_{b\Omega} = \phi\big|_{b\Omega} = f$. Thus $F$ is the required holomorphic extension of $f$ into $\Omega$.

□

## 1.4 Local extension of CR functions

In the extension theorem of Hartogs in section 1.3.4 and in the global extension results for CR functions in section 1.3.7, the geometry of the boundary plays no role (except implicitly in Theorem 1.3.37). Under a suitable hypothesis on the Levi form, there is a *local* extension phenomenon: holomorphic functions inside a domain extend across pseudoconcave parts of the boundary. It is reasonable to expect that CR functions on the boundary should extend analogously.

In the local context, one can dispense with the domain and consider a CR function on a small piece of a hypersurface. If the Levi form of a particular defining function has a negative eigenvalue, then the CR function extends holomorphically to one side of the hypersurface, and if the Levi form has a positive eigenvalue, then the CR function extends to the other side of the hypersurface. (If all the eigenvalues of the Levi form are equal to 0, then there need not exist an extension to either side. For example, on the hypersurface where $\operatorname{Re} z_1 = 0$, every real-valued, continuous function $f$ of $\operatorname{Im} z_1$ is a CR function; but if $f$ is the continuous boundary value of a holomorphic function, then $f$ has to be real-analytic because of the Schwarz reflection principle.)

To formulate a precise statement, suppose that $\rho$ is a class $C^4$ function in a neighborhood $U$ of the origin in $\mathbb{C}^n$ such that $\rho(0) = 0$ and $\nabla\rho \neq 0$ in $U$. (The natural differentiability hypothesis on $\rho$ is class $C^2$, but to avoid technical complications it is convenient to assume a little extra differentiability, as in Theorem 1.3.37.) Let $\Gamma$ denote the hypersurface $\{\, z \in U : \rho(z) = 0 \,\}$. To say that a vector $\sum_{j=1}^n t_j(\partial/\partial z_j)$ is tangent to $\Gamma$ at 0 means that $\sum_{j=1}^n t_j(\partial\rho/\partial z_j)(0) = 0$. The Levi form at the origin is the quadratic form acting on such tangent vectors as follows:

$$\sum_{j=1}^n \sum_{k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(0) t_j \bar{t}_k, \qquad \text{where } \sum_{j=1}^n t_j \frac{\partial \rho}{\partial z_j}(0) = 0.$$

**Theorem 1.4.1.** *Suppose $n \geq 2$. If, in the situation just described, the Levi form of $\rho$ has a negative eigenvalue at the origin, then there is a neighborhood $V$ of the origin such that for every CR function $f$ of class $C^4$ on $\Gamma$ there exists a continuous function $F$ on $\{\, z \in V : \rho(z) \geq 0 \,\}$ such that $F\big|_\Gamma = f$ and $F$ is holomorphic on $\{\, z \in V : \rho(z) > 0 \,\}$. Similarly, if the Levi form has a positive eigenvalue, then $f$ extends holomorphically to the side of $\Gamma$ where $\rho < 0$.*

**Exercise 1.4.2.** In the setting of the preceding theorem, show that if the Levi form has both a positive eigenvalue and a negative eigenvalue at the origin (which can happen only when $n \geq 3$), then there is a holomorphic function $F$ in some neighborhood of the origin such that $F\big|_\Gamma = f$. (The preceding theorem provides a continuous function $F$ that is holomorphic on each side of $\Gamma$, so what needs to be checked is that $F$ is holomorphic at points of $\Gamma$.)

*Proof of Theorem 1.4.1.* The strategy is similar to the proof of Theorem 1.3.37, except that one needs a local theorem about solving the $\overline{\partial}$-problem while keeping the support on one side of a hypersurface. The proof follows the exposition of Hörmander [10, proof of Theorem 2.6.13]. For a different proof that uses the approximation theorem of Baouendi and Treves [3] and the technique of analytic discs, see the book of Boggess [5, sections 15.1–15.2].

It suffices to consider the case when the Levi form has a negative eigenvalue. (Change the sign of the defining function $\rho$ if necessary.) It is convenient to normalize the defining function $\rho$ by making holomorphic coordinate changes in the following way.

Since the hypersurface $\Gamma$ is locally a graph over its tangent plane, one can choose local coordinates $(z_1, \ldots, z_n)$ in which $\Gamma$ is the set where $\operatorname{Im} z_n = \alpha(z_1, \ldots, z_{n-1}, \operatorname{Re} z_n) + O(|z|^3)$, where $\alpha$ is a quadratic form in $\operatorname{Re} z_1, \ldots, \operatorname{Re} z_{n-1}, \operatorname{Im} z_1, \ldots, \operatorname{Im} z_{n-1}$, and $\operatorname{Re} z_n$ with real coefficients. By making a local biholomorphic change of coordinates, one can eliminate the dependence of $\alpha$ on $\operatorname{Re} z_n$ (making a corresponding change in the form of the $O(|z|^3)$ remainder term). Indeed, if the $\operatorname{Re} z_n$ terms in $\alpha$ have the form $(\operatorname{Re} z_n)(\operatorname{Re} \sum_{j=1}^{n} \beta_j z_j)$, where the $\beta_j$ are complex constants, then the change of variables $z_j' = z_j$ when $1 \le j \le n-1$ and $z_n' = z_n - i z_n \sum_{j=1}^{n} \beta_j z_j$ (which is biholomorphic in a neighborhood of the origin because the Jacobian matrix at the origin is the identity matrix) eliminates those terms. Notice that terms of the form $(\operatorname{Im} z_n)(\operatorname{Im} \sum_{j=1}^{n} \beta_j z_j)$ can be included in the $O(|z|^3)$ remainder term because $\operatorname{Im} z_n = O(|z|^2)$ on $\Gamma$.

Consequently, one may assume that the quadratic form $\alpha$ is represented as the sum $\operatorname{Re} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} b_{jk} z_j z_k + \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} c_{jk} z_j \overline{z}_k$, where $(c_{jk})$ is a Hermitian-symmetric matrix. The biholomorphic change of variables $z_j' = z_j$ when $1 \le j \le n-1$ and $z_n' = z_n - i \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} b_{jk} z_j z_k$ reduces $\alpha$ to the form $\sum_{j=1}^{n-1} \sum_{k=1}^{n-1} c_{jk} z_j \overline{z}_k$. Now a unitary transformation in the first $n-1$ variables reduces $\alpha$ to the form $\sum_{j=1}^{n-1} a_j |z_j|^2$, where the $a_j$ are real numbers. In these new coordinates, the defining function $\rho(z_1, \ldots, z_n)$ may be taken to be $\operatorname{Im} z_n - \sum_{j=1}^{n-1} a_j |z_j|^2 + O(|z|^3)$. The hypothesis that the Levi form has a negative eigenvalue means that (at least) one of the numbers $a_j$ is positive. Suppose, without loss of generality, that $a_1 > 0$.

Lemma 1.3.39 provides an extension $\phi$ of the CR function $f$ to a neighborhood of the origin such that $\phi$ is class $C^2$ and $\overline{\partial}\phi = O(\rho^2)$. Choose a small positive number $\delta$ such that the polydisc of radius $\delta$ lies inside the neighborhood of the origin where $\phi$ is defined. In view of the continuity of $\rho$ and the continuity of the second derivatives of $\rho$, the following two properties can be achieved by shrinking $\delta$. First, whenever the point $(z_1, \ldots, z_n)$ lies in the polydisc centered at the origin of radius $\delta$, the function $z_1 \mapsto \rho(z_1, \ldots, z_n)$ is strictly superharmonic. Second, $\rho(z_1, 0, \ldots, 0) < 0$ when $0 < |z_1| \le \delta$. Fix $\delta$, and choose (again using the continuity of $\rho$) a smaller positive number $\epsilon$ such that (a) when $|z_1| = \delta$ and $\max(|z_2|, \ldots, |z_n|) \le \epsilon$, the value $\rho(z_1, \ldots, z_n) < 0$; and (b) when $|z_1| \le \delta$ and $\max(|z_2|, \ldots, |z_{n-1}|) \le \epsilon$, the value $\rho(z_1, z_2, \ldots, z_{n-1}, -i\epsilon) < 0$.

The proof is carried out on the closed polydisc of polyradius $(\delta, \epsilon, \ldots, \epsilon)$. The two main consequences of the choices of $\delta$ and $\epsilon$ are the following. First, the intersection of the polydisc with the set where $\rho < 0$ is a connected set. Indeed, this set contains (by construction) the points where $|z_1| = \delta$, so it suffices to show, for fixed values of $z_2, \ldots,$ and $z_n$, that the set $\{ z_1 : |z_1| \le \delta \text{ and } \rho(z_1, \ldots, z_n) < 0 \}$ is a connected subset of the closed disc of radius $\delta$. If this set had a component that did not intersect the boundary of the disc of radius $\delta$, then the continuous function $\rho$ would equal 0 on the boundary of the component. Consequently, $\rho$ would attain a (negative) minimum in the interior of the component, contradicting that $\rho$ is (by construction) superharmonic as a function of $z_1$. Second, there is a nonvoid open set in the space of variables $z_2, \ldots, z_n$ whose Cartesian product with $\{ z_1 \in \mathbb{C} : |z_1| \le \delta \}$ is contained in the intersection of the polydisc with the set where $\rho < 0$. This follows from item (b) in the construction in the preceding paragraph (by the continuity of $\rho$).

In a neighborhood of the closed polydisc with polyradius $(\delta, \epsilon, \ldots, \epsilon)$, define a $(0,1)$-form $G$ such that $G = \overline{\partial}\phi$ when $\rho \ge 0$ and $G = 0$ when $\rho \le 0$. Then $G$ is a $\overline{\partial}$-closed form of class $C^1$ because $\overline{\partial}\phi = O(\rho^2)$. Since $G(z_1, \ldots, z_n) = 0$ in a neighborhood of the set where $|z_1| = \delta$, one can extend $G$ to be identically 0 on the set where $|z_1| > \delta$ and $\max(|z_2|, \ldots, |z_n|) \le \epsilon$, and the extension is still class $C^1$. The goal now is to find a function $u$ such that $\overline{\partial}u = G$ in the polydisc centered at the origin with polyradius $(\delta, \epsilon, \ldots, \epsilon)$, and $u\big|_\Gamma = 0$. The idea is to apply the Cauchy formula (1.1) in the first coordinate.

Let $G_j(w)$ denote the $d\overline{w}_j$ component of the $(0,1)$-form $G(w)$, and define $u$ in the polydisc as follows:

$$u(z_1, \ldots, z_n) := -\frac{1}{\pi} \int_{\substack{|w_1|<\delta \\ \text{or } w_1 \in \mathbb{C}}} \frac{G_1(w_1, z_2, \ldots, z_n)}{w_1 - z_1}\, d\mathsf{Area}_{w_1}.$$

The same change-of-variables argument as in the proof of Theorem 1.3.6 shows that

$$\frac{\partial u}{\partial \overline{z}_1} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{(\partial G_1/\partial \overline{w}_1)(w_1, z_2, \ldots, z_n)}{w_1 - z_1}\, d\mathsf{Area}_{w_1}.$$

The one-dimensional Cauchy integral formula (1.1) now implies that $\partial u/\partial \overline{z}_1 = G_1$. Moreover, differentiating under the integral sign and using that $G$ is $\overline{\partial}$-closed implies for $k$ different from 1 that

$$\frac{\partial u}{\partial \overline{z}_k} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial G_1/\partial \overline{z}_k}{w_1 - z_1}\, d\mathsf{Area}_{w_1} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial G_k/\partial \overline{w}_1}{w_1 - z_1}\, d\mathsf{Area}_{w_1}.$$

Again invoking (1.1) shows that $\partial u/\partial \overline{z}_k = G_k$. Thus $\overline{\partial}u = G$ as claimed.

In particular, $u$ is a holomorphic function in intersection of the polydisc with the set where $\rho < 0$. As observed above, this part of the polydisc is a connected set; moreover, this part of the polydisc contains an open subset that is fibered by discs of radius $\delta$ in the first coordinate direction. The function $u$ is identically equal to zero on this open subset (because $G(w_1, z_2, \ldots, z_n) = 0$ on the whole disc where $|w_1| < \delta$). By the identity theorem for holomorphic functions, the function $u$ is identically equal to zero on the intersection of the polydisc with the set where $\rho < 0$. By continuity, the function $u = 0$ on $\Gamma$.

If $F := \phi - u$, then $F$ is class $C^1$, and $F\big|_\Gamma = \phi\big|_\Gamma = f$. On the set where $\rho > 0$, one has that $\overline{\partial} F = \overline{\partial}\phi - \overline{\partial} u = 0$. Thus $F$ is the required local holomorphic extension of $f$ to the side of $\Gamma$ where $\rho > 0$. □

## 1.5 The Cauchy–Fantappiè integral representation

The Bochner–Martinelli integral belongs to a family of integral representations that the famous French mathematician Jean Leray (born 7 November 1906, died 10 November 1998) dubbed "Cauchy–Fantappiè integrals" after the work of the Italian mathematician Luigi Fantappiè (born 15 September 1901, died 28 July 1956). The first instance of the terminology "Cauchy–Fantappiè" seems to be in Leray's long memoir [24] from 1959, but he published some of the ideas in two short notes [22] and [23] in 1956.

Although the Bochner–Martinelli integral has a universal kernel, the kernel of the Cauchy–Fantappiè integral needs to be constructed specially for each domain. In favorable cases, such as the case of convex domains to be considered later on, one can exhibit an explicit Cauchy–Fantappiè kernel that is holomorphic in the free variable. Consequently, the elaborate formalism for integral representations more general than the Bochner–Martinelli integral has a significant payoff.

A famous application of the Cauchy–Fantappiè formalism at the end of the 1960s was the construction for strongly pseudoconvex domains of a nearly explicit integral representation formula that is holomorphic in the free variable. The construction was carried out independently by E. Ramírez [27] (the published article is based on his dissertation at the University of Göttingen) and by G. M. Henkin [9].

**Definition 1.5.1.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with class $C^1$ boundary. Suppose $g\colon \mathsf{neighborhood}(b\Omega)\times\Omega \to \mathbb{C}^n$ is a vector-valued function with the following two properties: (a) for each fixed $z$ in $\Omega$, the function $w \mapsto g(w,z)$ is class $C^1$; and (b) the scalar product

$$\langle g(w,z), \overline{w} - \overline{z}\rangle \qquad \text{is nonzero when } w \text{ is in } b\Omega \text{ and } z \text{ is in } \Omega. \tag{1.33}$$

The angle brackets indicate the usual scalar product on $\mathbb{C}^n$: namely, $\langle g(w,z), \overline{w} - \overline{z}\rangle = \sum_{k=1}^n g_k(w,z)(w_k - z_k)$. (Be aware that in this context some authors use the angle brackets to denote the corresponding sum without the conjugation in the second factor, since that notation is consistent both with the action of a differential form on a vector field and with the action of a linear functional on a function.) The *Cauchy–Fantappiè kernel* generated by $g$ is the differential form of bidegree $(n, n-1)$ given by the expression

$$c_n \sum_{j=1}^n \frac{(-1)^{j-1} g_j(w,z)}{\langle g(w,z), \overline{w} - \overline{z}\rangle^n}\, \overline{\partial}_w g[j](w,z) \wedge dw, \tag{1.34}$$

where, as in Definition 1.3.1, the dimensional constant $c_n$ equals $(-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n$, the $(0, n-1)$-form $\overline{\partial}_w g[j]$ equals $\overline{\partial}_w g_1 \wedge \cdots \wedge \overline{\partial}_w g_{j-1} \wedge \overline{\partial}_w g_{j+1} \wedge \cdots \wedge \overline{\partial}_w g_n$, and the $(n, 0)$-form $dw$ equals $dw_1 \wedge \cdots \wedge dw_n$.

In (1.34), one could write $d_w g[j](w, z)$ instead of $\overline{\partial}_w g[j](w, z)$ (for degree reasons, since the kernel already contains $dw_1 \wedge \cdots \wedge dw_n$). Notice that one obtains the Bochner–Martinelli kernel (1.15) as a special case of (1.34) by taking $g(w, z)$ equal to $\overline{w} - \overline{z}$.

It is not essential that the function $g$ be defined for $w$ in a *neighborhood* of $b\Omega$; one really needs $g$ to be defined only on $b\Omega \times \Omega$, for $g$ can be extended in an arbitrary way from $b\Omega$ to a neighborhood of $b\Omega$ (as a class $C^1$ function). The restriction of (1.34) to $b\Omega$ is independent of the extension of $g$, and since the plan is to integrate over $b\Omega$, the ultimate formula will be independent of the particular extension of $g$.

One often sees the definition of the Cauchy–Fantappiè kernel written not with (1.33) but instead with the more restrictive condition that

$$\langle g(w, z), \overline{w} - \overline{z} \rangle = 1 \qquad \text{for all } w \text{ in } b\Omega \text{ and } z \text{ in } \Omega. \tag{1.35}$$

There is little loss of generality in imposing this extra restriction, for one can replace the function $g(w, z)$ by the function $g(w, z)/\langle g(w, z), \overline{w} - \overline{z} \rangle$. The following exercise implies that this rescaled function generates *exactly the same* Cauchy–Fantappiè kernel as the kernel that $g$ itself generates.

**Exercise 1.5.2.** Show that if $\lambda$ is a class $C^1$ function with no zeroes on $b\Omega$, and $G = \lambda g$, then the Cauchy–Fantappiè kernel generated by $G$ equals the Cauchy–Fantappiè kernel generated by $g$.

**Exercise 1.5.3.** Let $\gamma$ denote the $(1, 0)$-form $\sum_{j=1}^n g_j(w, z)\, dw_j$. Show that

$$\frac{1}{(2\pi i)^n} \cdot \frac{\gamma \wedge (\overline{\partial}_w \gamma)^{n-1}}{\langle g(w, z), \overline{w} - \overline{z} \rangle^n}$$

is an equivalent expression for the Cauchy–Fantappiè kernel (1.34). In particular, if $g$ satisfies the condition (1.35), then the kernel (1.34) reduces to

$$\frac{1}{(2\pi i)^n}\, \gamma \wedge (\overline{\partial}_w \gamma)^{n-1}. \tag{1.36}$$

**Remark 1.5.4.** Walter Koppelman [16] observed that one can use a determinant to rewrite the expression (1.34) for the Cauchy–Fantappiè kernel. When condition (1.35) is in force, the Cauchy–Fantappiè kernel can be expressed as follows:

$$\frac{1}{(2\pi i)^n} \det \begin{vmatrix} g_1\, dw_1 & \overline{\partial} g_1 \wedge dw_2 & \dots & \overline{\partial} g_1 \wedge dw_n \\ g_2\, dw_1 & \overline{\partial} g_2 \wedge dw_2 & \dots & \overline{\partial} g_2 \wedge dw_n \\ \vdots & \vdots & \ddots & \vdots \\ g_n\, dw_1 & \overline{\partial} g_n \wedge dw_2 & \dots & \overline{\partial} g_n \wedge dw_n \end{vmatrix}. \tag{1.37}$$

Indeed, the definition of a determinant shows that (1.37) equals the following sum over permutations $\sigma$ of the $n$ indices $1, \ldots, n$:

$$\frac{1}{(2\pi i)^n} \sum_{\sigma} \operatorname{sign}(\sigma) g_{\sigma(1)}\, dw_1 \wedge \overline{\partial} g_{\sigma(2)} \wedge dw_2 \wedge \cdots \wedge \overline{\partial} g_{\sigma(n)} \wedge dw_n,$$

where $\operatorname{sign}(\sigma)$ equals $+1$ if $\sigma$ is a product of an even number of transpositions and $-1$ if $\sigma$ is a product of an odd number of transpositions. On the other hand, the differential form (1.36) equals the sum

$$\frac{1}{(2\pi i)^n} \sum_{\sigma} g_{\sigma(1)}\, dw_{\sigma(1)} \wedge \overline{\partial} g_{\sigma(2)} \wedge dw_{\sigma(2)} \wedge \cdots \wedge \overline{\partial} g_{\sigma(n)} \wedge dw_{\sigma(n)}.$$

Reordering the differentials shows that the preceding two expressions are indeed equal.

The determinant (1.37) is slightly modified from Koppelman's formulation in order to suppress a technical complication. In considering determinants with entries from a noncommutative ring, one has to be careful about the order of terms in the products that arise. (For some discussion of determinants with differential forms as entries, see [35, §24.3, pp. 208–210] and [2, pp. 5–7].) In the preceding determinant, one does not have to worry about noncommutativity, for differential forms of degree 2 commute with differential forms of arbitrary degree.

**Theorem 1.5.5.** *If $\Omega$ is a bounded domain in $\mathbb{C}^n$ with class $C^1$ boundary, then every continuous function $f$ on the closure of $\Omega$ that is holomorphic in $\Omega$ is reproduced by integration against any Cauchy–Fantappiè kernel* (1.34)*: namely,*

$$f(z) = c_n \int_{b\Omega} f(w) \sum_{j=1}^{n} \frac{(-1)^{j-1} g_j(w,z)}{\langle g(w,z), \overline{w} - \overline{z}\rangle^n}\, \overline{\partial}_w g[j](w,z) \wedge dw \qquad \text{when } z \in \Omega. \tag{1.38}$$

Notice that when $n = 1$, the equation (1.38) reduces to the usual one-dimensional Cauchy integral formula.

**Example 1.5.6.** When $\Omega$ is the unit ball in $\mathbb{C}^n$, one can choose a particularly simple expression for $g$: namely, $g(w,z) = \overline{w}$. If $w$ is in the boundary of the unit ball, then $\sum_{j=1}^{n} g_j(w,z)(w_j - z_j) = 1 - \langle z, w\rangle$, and $1 - \langle z, w\rangle \neq 0$ when $z$ is inside the ball. Notice that this choice of $g$ is holomorphic in $z$ simply because $g$ is independent of $z$.

The corresponding Cauchy–Fantappiè kernel (1.34) then becomes

$$c_n \sum_{j=1}^{n} \frac{(-1)^{j-1} \overline{w}_j}{(1 - \langle z, w\rangle)^n}\, d\overline{w}[j] \wedge dw.$$

In view of (1.21), the Cauchy–Fantappiè reproducing formula (1.38) for the ball can be rewritten as follows:

$$f(z) = \frac{(n-1)!}{2\pi^n} \int_{b\Omega} \frac{f(w)}{(1-\langle z,w\rangle)^n}\, d\mathsf{SurfaceArea}_w.$$

The kernel function in this integral is the Szegő kernel function for the unit ball in $\mathbb{C}^n$. In addition to being holomorphic in $z$, the Szegő kernel function is conjugate symmetric in the variables $z$ and $w$. (In fancier language, the integral operator corresponding to integration against the Szegő kernel function is a self-adjoint operator on $L^2(b\Omega, d\mathsf{SurfaceArea})$.)

*Proof of Theorem 1.5.5.* There are two somewhat different proofs of the theorem, both of which are enlightening. The common theme of the two proofs is to relate the Cauchy–Fantappiè integral to the Bochner–Martinelli integral, but one proof passes to the interior while the other proof stays on the boundary.

In both proofs, there is no loss of generality in assuming that $f$ is holomorphic in a neighborhood of the closure of $\Omega$: one can work on an increasing sequence of subdomains that exhaust $\Omega$ (say sublevel sets of a defining function) and pass to the limit. Moreover, it suffices to establish (1.38) for a specified $z$ in $\Omega$. If one understands that $z$ is fixed from now on, then one can simplify the notation by dropping the subscript $w$ on the differential operators $d$ and $\overline{\partial}$.

The first proof uses Exercise 1.5.3. Suppose that $g(w,z)$ has been extended to $\mathbb{C}^n$ as a class $C^1$ function. By hypothesis, the set of points $w$ for which $\sum_{j=1}^n g_j(w,z)(w_j - z_j) \neq 0$ is an open set containing $b\Omega$. The initial claim is that on this open set, the Cauchy–Fantappiè kernel (1.34) is a closed differential form. In verifying this claim, one can assume in view of Exercise 1.5.2 that $\sum_{j=1}^n g_j(w,z)(w_j - z_j) = 1$. Consequently, by Exercise 1.5.3, one needs to check that $d[\gamma \wedge (\overline{\partial}\gamma)^{n-1}] = 0$, where $\gamma = \sum_{j=1}^n g_j(w,z)\, dw_j$. For degree reasons, it is equivalent to check that $(\overline{\partial}\gamma)^n = 0$. Since $\overline{\partial}\gamma = \sum_{j=1}^n \overline{\partial} g_j \wedge dw_j$, one sees that $(\overline{\partial}\gamma)^n = n!\,(\overline{\partial} g_1 \wedge dw_1) \wedge \cdots \wedge (\overline{\partial} g_n \wedge dw_n)$. But the differentials $\overline{\partial} g_1, \ldots, \overline{\partial} g_n$ are linearly dependent, since $\sum_{j=1}^n g_j(w,z)(w_j - z_j) = 1$, so $(\overline{\partial}\gamma)^n = 0$ as claimed.

The proof actually uses the closedness not of the Cauchy–Fantappiè form generated by $g$ but of the Cauchy–Fantappiè form generated by a function constructed from $g$ as follows. As indicated above, one may assume that $\sum_{j=1}^n g_j(w,z)(w_j - z_j) = 1$ in a neighborhood of $b\Omega$. Let $\chi$ be a smooth cut-off function supported in this neighborhood such that $0 \le \chi \le 1$, the function $\chi$ is identically equal to 1 in a smaller neighborhood of $b\Omega$, and $\chi$ is identically equal to 0 in a neighborhood of the specified point $z$. Let $G$ be the vector-valued function defined by the property that $G(w) = \chi(w) g(w,z) + (1-\chi(w))(\overline{w} - \overline{z})$. On the set where $\chi(w) = 1$, the function $G$ inherits from $g$ the property that $\sum_{j=1}^n G_j(w)(w_j - z_j) = 1$. When $0 < \chi(w) < 1$, one has that $\sum_{j=1}^n G_j(w,z)(w_j - z_j) = \chi(w) + (1-\chi(w))|w-z|^2 > 0$.

On the set where $\chi(w) = 0$, one has that $\sum_{j=1}^{n} G_j(w, z)(w_j - z_j) = |w - z|^2$, which is equal to 0 if and only if $w = z$. Consequently, the Cauchy–Fantappiè form generated by $G$ is a well-defined closed differential form on $\mathsf{closure}(\Omega) \setminus \{z\}$.

If $f$ is a holomorphic function, then $f(w)$ times the Cauchy–Fantappiè form generated by $G$ is still a closed form on $\mathsf{closure}(\Omega) \setminus \{z\}$. Consequently, the integral of that product over $b\Omega$ equals the integral over the boundary of a small ball centered at $z$. On that small neighborhood of $z$, the function $G(w)$ is equal to $\overline{w} - \overline{z}$ by construction. Therefore the integral of $f$ times the Cauchy–Fantappiè form generated by $G$ is equal to the integral over the boundary of a small ball centered at $z$ of $f$ times the Bochner–Martinelli kernel, which equals $f(z)$ by the Bochner–Martinelli formula (1.19). By construction, the restriction to $b\Omega$ of the Cauchy–Fantappiè kernel generated by $G$ equals the Cauchy–Fantappiè kernel generated by $g$. Consequently, the Cauchy–Fantappiè integral representation (1.38) does hold. This completes the first proof of the theorem.

The second proof, which is due to G. M. Henkin [9], elegantly illustrates the power of the language of differential forms. Consider in $\mathbb{C}^n \times \mathbb{C}^n$ the complex submanifold $\Gamma$ of complex dimension $2n-1$ consisting of points $(\zeta, \eta)$ such that $\zeta$ lies in an open neighborhood of the closure of $\Omega$, and $\langle \eta, \overline{\zeta} - \overline{z} \rangle = 1$. The differential form $f(\zeta) \sum_{j=1}^{n} (-1)^{j-1} \eta_j \, d\eta[j] \wedge d\zeta$ has the property that its restriction to $\Gamma$ is a closed form on $\Gamma$. Indeed, the exterior derivative of $f(\zeta) \sum_{j=1}^{n} (-1)^{j-1} \eta_j \, d\eta[j] \wedge d\zeta$ computed in $\mathbb{C}^n \times \mathbb{C}^n$ equals $n f(\zeta) \, d\eta \wedge d\zeta$ (since $f$ is holomorphic), which is a form of bidegree $(2n, 0)$, and the restriction of this differential form to the $(2n-1)$-dimensional complex manifold $\Gamma$ equals 0 for degree reasons.

Consider inside $\Gamma$ the two submanifolds $\gamma_1$ and $\gamma_2$ of real dimension $2n - 1$ defined as follows:

$$\begin{aligned} \gamma_1 &= \{ (\zeta, \eta) \in \Gamma : \zeta \in b\Omega \text{ and } \eta = (\overline{\zeta} - \overline{z})/|\zeta - z|^2 \}, \\ \gamma_2 &= \{ (\zeta, \eta) \in \Gamma : \zeta \in b\Omega \text{ and } \eta = g(\zeta, z)/\langle g(\zeta, z), \overline{\zeta} - \overline{z} \rangle \}. \end{aligned}$$

By the defining property (1.33) of Cauchy–Fantappiè integrals, the denominators in the definitions of $\gamma_1$ and $\gamma_2$ are nonzero, so $\gamma_1$ and $\gamma_2$ are well defined. The property that $\langle \eta, \overline{\zeta} - \overline{z} \rangle = 1$ evidently holds both for points in $\gamma_1$ and for points in $\gamma_2$, so $\gamma_1$ and $\gamma_2$ do lie in $\Gamma$. Moreover, there is a simple homotopy in $\Gamma$ between $\gamma_1$ and $\gamma_2$ determined by setting $\eta$ equal to the expression

$$t \frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^2} + (1 - t) \frac{g(\zeta, z)}{\langle g(\zeta, z), \overline{\zeta} - \overline{z} \rangle}, \qquad \text{where } 0 \le t \le 1.$$

Consequently,

$$\int_{\gamma_1} f(\zeta) \sum_{j=1}^{n} (-1)^{j-1} \eta_j \, d\eta[j] \wedge d\zeta = \int_{\gamma_2} f(\zeta) \sum_{j=1}^{n} (-1)^{j-1} \eta_j \, d\eta[j] \wedge d\zeta,$$

since the common integrand is a closed differential form on $\Gamma$.

Pulling back the integral over $\gamma_1$ into $\mathbb{C}^n$ by using the parametrization of $\gamma_1$ produces the Bochner–Martinelli integral of $f$ (in view of Exercise 1.5.2), which equals $f(z)$ by the Bochner–Martinelli formula (1.19). Similarly, pulling back the integral over $\gamma_2$ into $\mathbb{C}^n$ by using the parametrization of $\gamma_2$ produces the Cauchy–Fantappiè integral of $f$ generated by $g$. Therefore, as claimed, the Cauchy–Fantappiè integral does equal $f(z)$. $\square$

### 1.5.1 The Cauchy–Fantappiè integral for convex domains

Example 1.5.6 shows that in the case of the ball, one can exhibit an explicit Cauchy–Fantappiè kernel that is holomorphic in the free variable. One can write down a similar kernel more generally in the case of convex domains.

Since convexity is a condition that depends on second derivatives, the natural setting is a bounded domain in $\mathbb{C}^n$ defined by a class $C^2$ real-valued function $\rho$ (whose gradient is nonzero on the boundary of the domain). If one writes $z = x + iy$ and $w = u + iv$, then

$$2\,\mathrm{Re}\sum_{j=1}^{n} \frac{\partial\rho}{\partial w_j}(w)(w_j - z_j) = \sum_{j=1}^{n} \frac{\partial\rho}{\partial u_j}(u,v)(u_j - x_j) + \frac{\partial\rho}{\partial v_j}(u,v)(v_j - y_j).$$

The right-hand side equals the real scalar product of the real gradient vector of $\rho$ with the real vector representing the difference $w - z$. The convexity of $\Omega$ implies that this scalar product is nonzero when $w \in b\Omega$ and $z \in \Omega$. Therefore one gets a function $g$ satisfying the condition (1.33) by setting $g_j$ equal to $\partial\rho/\partial w_j$. In view of Exercise 1.5.3, one has the following Cauchy–Fantappiè kernel for a convex domain in $\mathbb{C}^n$:

$$\frac{1}{(2\pi i)^n} \cdot \frac{\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}(w)}{\left\langle \frac{\partial\rho}{\partial w}(w), \overline{w} - \overline{z} \right\rangle^n}. \tag{1.39}$$

As in the case of the ball, this kernel is holomorphic in the free variable $z$ because the numerator is independent of $z$ and the denominator is holomorphic in $z$ (since the scalar product conjugates the second factor).

Rewriting (1.39) in terms of surface area measure produces an illuminating integral-representation formula that involves the complex geometry of the boundary. The statement of the formula uses the following notation. When $w \in b\Omega$, let $E_\rho(w)$ denote the determinant of the Levi form of $\rho$ at $w$ (where the Levi form is viewed as a Hermitian quadratic form on the complex tangent space). In other words, $E_\rho(w)$ equals the product of the $n-1$ eigenvalues of the Levi form of $\rho$ at $w$. The following result is due to Lev Aĭzenberg [1].

**Theorem 1.5.7.** *Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$ with class $C^2$ defining function $\rho$. If $f$ is a continuous function on the closure of $\Omega$ that is holomorphic in $\Omega$, then*

$$f(z) = \frac{(n-1)!}{4\pi^n} \int_{b\Omega} f(w) \frac{E_\rho(w)|\nabla\rho(w)|}{\left\langle \frac{\partial\rho}{\partial w}(w), \overline{w} - \overline{z} \right\rangle^n} \, d\mathsf{SurfaceArea}_w \qquad \text{when } z \in \Omega, \tag{1.40}$$

*where $E_\rho$ is the determinant of the Levi form of $\rho$.*

*Proof.* In view of Theorem 1.5.5, one has to show that $\frac{1}{4}E_\rho|\nabla\rho|^2 \, d\mathsf{SurfaceArea}$ equals the restriction to $b\Omega$ of the differential form

$$(-1)^{n(n-1)/2}(2i)^{-n}|\nabla\rho(w)| \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial\rho}{\partial w_j} \, \overline{\partial} \left( \frac{\partial\rho}{\partial w_1} \right) \wedge \cdots [j] \cdots \wedge \overline{\partial} \left( \frac{\partial\rho}{\partial w_n} \right) \wedge dw, \tag{1.41}$$

where the notation $[j]$ indicates that the $j$th term is omitted.

Temporarily fixing both an index $j$ and an index $k$, consider how $d\overline{w}[k]$ terms arise in the differential form

$$\overline{\partial} \left( \frac{\partial\rho}{\partial w_1} \right) \wedge \cdots [j] \cdots \wedge \overline{\partial} \left( \frac{\partial\rho}{\partial w_n} \right). \tag{1.42}$$

There is one such term corresponding to each permutation $\sigma$ of the set $S_k$ of $n-1$ indices $\{1, \ldots, n\} \setminus \{k\}$, and each term comes with a plus or minus sign equal to the sign of the permutation $\sigma$. Now letting $k$ vary shows that (1.42) equals

$$\sum_{k=1}^{n} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \frac{\partial^2\rho}{\partial w_1 \partial \overline{w}_{\sigma(1)}} \cdots [j] \cdots \frac{\partial^2\rho}{\partial w_n \partial \overline{w}_{\sigma(n)}} \, d\overline{w}[k].$$

One finds from Exercises 1.2.2 and 1.2.3 that the restriction of $d\overline{w}[k] \wedge dw$ to the boundary of $\Omega$ equals $(-1)^{n(n-1)/2}(2i)^n(-1)^{k-1}|\nabla\rho|^{-1}\partial\rho/\partial\overline{w}_k$ times the surface area measure. Consequently, the restriction of the differential form (1.41) to the boundary of $\Omega$ equals

$$\sum_{j=1}^{n} (-1)^{j-1} \frac{\partial\rho}{\partial w_j} \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial\rho}{\partial \overline{w}_k} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \frac{\partial^2\rho}{\partial w_1 \partial \overline{w}_{\sigma(1)}} \cdots [j] \cdots \frac{\partial^2\rho}{\partial w_n \partial \overline{w}_{\sigma(n)}} \, d\mathsf{SurfaceArea}.$$

By staring hard at the preceding expression, one can recognize the factor multiplying the surface area element as being equal to

$$-\det \begin{vmatrix} 0 & \frac{\partial\rho}{\partial w_1} & \cdots & \frac{\partial\rho}{\partial w_n} \\ \frac{\partial\rho}{\partial \overline{w}_1} & \frac{\partial^2\rho}{\partial w_1 \partial \overline{w}_1} & \cdots & \frac{\partial^2\rho}{\partial w_n \partial \overline{w}_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\rho}{\partial \overline{w}_n} & \frac{\partial^2\rho}{\partial w_1 \partial \overline{w}_n} & \cdots & \frac{\partial^2\rho}{\partial w_n \partial \overline{w}_n} \end{vmatrix}. \tag{1.43}$$

The sum on $j$ corresponds to expanding the $(n+1) \times (n+1)$ determinant across the top row, and the sum on $k$ corresponds to expanding an $n \times n$ minor along the first column. The sum on permutations $\sigma$ corresponds to expanding the final $(n-1) \times (n-1)$ minor. The extra minus sign in front appears because the factor $(-1)^{j-1}$ has the wrong parity for the initial expansion across the top row.

The determinant in (1.43) is sometimes called the *Levi determinant*, because when $n = 2$ it equals an expression that Levi introduced to characterize pseudoconvexity through differential geometry. Wilhelm Wirtinger (born 15 July 1865, died 15 January 1945) seems to be the person who first wrote down the corresponding determinant in arbitrary dimension [38, p. 363]. Subsequently Erich Kähler (born 16 January 1906, died 31 May 2000) considered the determinant in a different context in his fundamental article [11] (reprinted in [12]) that introduced the notion later known by the name "Kähler metric".

In principle, one should be able to compute the Levi determinant (1.43) in terms of the eigenvalues of the Levi form by clever row and column operations, but there is an easier method. In deriving (1.43), one could have started not with the expression (1.41) but with a constant times $|\nabla\rho|\, \partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}$. The latter expression is unchanged if the coordinates are transformed by a unitary transformation. Indeed, the differential form $\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}$ is defined in a coordinate-free manner, and the length of $\nabla\rho$ is preserved by a unitary transformation. Consequently, to evaluate the Levi determinant (1.43) at a particular boundary point, one may choose convenient coordinates. Suppose that the first $(n-1)$ coordinate directions are tangential to the boundary at the given point and that the Levi form is diagonal at the given point. Expanding the Levi determinant across the first row and then along the first column shows that (1.43) equals

$$\left|\frac{\partial\rho}{\partial w_n}\right|^2 E_\rho \qquad \text{or, equivalently,} \qquad \tfrac{1}{4} E_\rho |\nabla\rho|^2.$$

That calculation establishes the claim stated at the beginning of the proof. □

**Exercise 1.5.8.** The preceding proof implicitly assumes that the dimension $n$ is at least 2. Show that when $n = 1$, equation (1.40) reduces to the ordinary one-dimensional Cauchy integral formula. (By the standard convention that an empty product is equal to 1, one has that $E_\rho = 1$ when $n = 1$.)

**Exercise 1.5.9.** Although the number of positive eigenvalues of the Levi form is independent of the choice of defining function, the eigenvalues themselves do depend on the defining function. Show that nonetheless the kernel in (1.40) depends only on $\Omega$ and not on the choice of the defining function $\rho$. Thus one can speak of "the" Cauchy integral for convex domains.

## The case of strongly pseudoconvex domains (sketch)

A strongly pseudoconvex domain is locally biholomorphically equivalent to a strongly convex domain. Therefore one expects that the preceding machinery should yield a reasonably explicit integral representation formula with a kernel that depends holomorphically on the free variable in the case of a bounded, strongly pseudoconvex domain. The technical complication is passing from a local kernel to a global kernel while preserving holomorphicity in the free variable.

There are two approaches to implementing this scheme, both of which are merely sketched here to indicate the ideas. To see details, consult the book of Range [29, Chapter VII, sections 1 and 3] or the original papers. Both approaches use the solvability of the $\overline{\partial}$-problem on (a neighborhood of the closure of) a strongly pseudoconvex domain to make the passage from local information to global information.

Both methods also start with the quadratic "Levi polynomial" of a defining function $\rho$. By Taylor's theorem, the positive definiteness of the complex Hessian of $\rho$ implies that if $w$ is a point on or near the boundary of $\Omega$, and $z$ is a point in a small neighborhood of $w$, then

$$\begin{aligned}2\operatorname{Re}\sum_{j=1}^{n}\frac{\partial\rho}{\partial w_j}(w)(w_j-z_j)-\operatorname{Re}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial w_j\partial w_k}(w)(w_j-z_j)(w_k-z_k)&\\ \geq \rho(w)-\rho(z)+(\text{positive constant})|w-z|^2&.\end{aligned}$$

Since the boundary of the bounded domain $\Omega$ is compact, the positive constant and the size of the neighborhood can be chosen to be independent of $w$. If the point $w$ is on the boundary of $\Omega$, and the nearby point $z$ is in the closure of $\Omega$ but not equal to $w$, then the right-hand side is positive. Moreover, one can even allow the point $z$ to go outside the closure of $\Omega$ as long as $z$ stays in a "tomato-shaped" region with the stem pit at $w$.

Thus a *local* candidate for the function $g_j(w,z)$ needed to guarantee a nonzero denominator in the Cauchy–Fantappiè form (1.34) is

$$\frac{\partial\rho}{\partial w_j}(w)-\frac{1}{2}\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial w_j\partial w_k}(w)(w_k-z_k),$$

an expression that is indeed holomorphic in the free variable $z$. A natural way to get a *global* function $g_j$ is to patch this candidate with the function that enters into the Bochner–Martinelli kernel. Take a cut-off function $\chi$ of one real variable that is identically equal to 1 in a neighborhood of the origin and that has small support. Define a global $g_j(w,z)$ as

$$\chi(|w-z|)\left(\frac{\partial\rho}{\partial w_j}(w)-\frac{1}{2}\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial w_j\partial w_k}(w)(w_k-z_k)\right)+(1-\chi(|w-z|))(\overline{w}_j-\overline{z}_j).$$

This global function does generate a Cauchy–Fantappiè form that reproduces holomorphic functions on $\Omega$, but the kernel is not globally holomorphic in $z$.

At this point the two approaches diverge. The original method due independently to Henkin [9] and to Ramírez [27] corrects at the level of functions to obtain a new Cauchy–Fantappiè form that is globally holomorphic in the free variable $z$. A later method due jointly to Norberto Kerzman and Elias M. Stein [15] corrects at the level of forms to obtain an integral representation with a kernel that is not a Cauchy–Fantappiè kernel but that differs from a Cauchy–Fantappiè by a relatively tame non-explicit kernel.

In the Kerzman-Stein approach, one starts with the Cauchy–Fantappiè kernel indicated above. For a fixed $w$ in the boundary, the kernel has a singularity as $z$ tends to $w$. Since the kernel is holomorphic with respect to $z$ for $z$ near $w$, taking $\overline{\partial}_z$ gives a globally defined kernel that is smooth (identically equal to 0) for $z$ near $w$. Then one solves a $\overline{\partial}$-problem to correct the kernel to make it globally holomorphic, and the correction term is non-singular for $z$ near $w$. Of course one has to do this uniformly with respect to $w$. The remaining key point is to see that the correction term does not destroy the reproducing property, since the corrected kernel is no longer a Cauchy–Fantappiè kernel. One has to show that the integral over the boundary of $\Omega$ of the correction term times a holomorphic function $f$ equals 0. Since the solution of the $\overline{\partial}_z$-problem is given by a linear operator, the problem reduces to showing that the integral over the boundary of $\Omega$ of a holomorphic function times $\overline{\partial}_z$ of a Cauchy–Fantappiè kernel equals 0. A proposition analogous to Lemma 1.3.8 shows that $\overline{\partial}_z$ of any Cauchy–Fantappiè kernel is $\overline{\partial}_w$-exact, and so property (1.28) of CR-functions finishes the argument.

In the Henkin-Ramírez approach, one initially has the condition (1.33): a certain function of $w$ and $z$ is nonvanishing when $w \neq z$. Then one solves a $\overline{\partial}$-problem to correct the function to a new function $\Phi(w, z)$ such that $\Phi(w, w) = 0$, and $\Phi$ is holomorphic in $z$ and nonvanishing when $w \neq z$. To get back to a vector-function needed to generate a Cauchy–Fantappiè kernel, one solves a "division problem" to write $\Phi(w, z) = \sum_{j=1}^{n} \phi_j(w, z)(w_j - z_j)$. That step uses a result known as "Hefer's lemma" or "Hefer's theorem" after the 1940 Münster dissertation of Hans Hefer, an excerpt of which was published posthumously [8]. Apparently Hefer was killed in action in World War II the year after defending his dissertation.[1]

**Theorem 1.5.10** (Hefer). *If $f$ is a holomorphic function in a domain of holomorphy $\Omega$ in $\mathbb{C}^n$, then there exist holomorphic functions $g_1, \ldots, g_n$ in $\Omega \times \Omega$ such that*

$$f(w) - f(z) = \sum_{j=1}^{n} g_j(w, z)(w_j - z_j) \qquad \textit{for all } z \textit{ and } w \textit{ in } \Omega.$$

[1] A footnote by H. Behnke and K. Stein to Hefer's posthumous article says, "Der Verfasser ist 1941 im Osten gefallen."

Hefer's theorem is easy when the dimension $n$ equals 1. In that case,

$$g_1(w,z) = \begin{cases} \dfrac{f(w)-f(z)}{w-z}, & \text{when } w \neq z, \\ f'(w), & \text{when } w = z. \end{cases}$$

Evidently $g_1$ is a continuous function that is holomorphic in each variable separately, so $g_1$ is a holomorphic function of the two variables jointly.

In higher dimensions, there is no simple, general formula for producing a Hefer decomposition, but the special case of a polynomial function $f$ is easy to handle. By changing one variable at a time, one can write

$$\begin{aligned} f(w_1,\dots,w_n) - f(z_1,\dots,z_n) = {} & f(w_1,\dots,w_n) - f(z_1,w_2,\dots,w_n) \\ & + f(z_1,w_2,\dots,w_n) - f(z_1,z_2,w_3,\dots,w_n) \\ & + \cdots + f(z_1,\dots,z_{n-1},w_n) - f(z_1,\dots,z_n). \end{aligned}$$

(The intermediate terms are evaluated at points that need not be in $\Omega$, so this decomposition makes sense only when $f$ is an entire function.) One achieves the conclusion of Hefer's theorem by factoring $w_k - z_k$ out of the $k$th term on the right-hand side for each $k$ between 1 and $n$. As a consequence of this special case, one can obtain Hefer's theorem for those domains (even nonpseudoconvex ones) in which polynomials are dense in the holomorphic functions.

*Proof of Hefer's theorem.* The first idea in the proof is to increase the dimension: namely, view the difference $f(w) - f(z)$ as a holomorphic function on the domain of holomorphy $\Omega \times \Omega$ in $\mathbb{C}^{2n}$. This holomorphic function vanishes on the $n$-dimensional complex plane in $\mathbb{C}^{2n}$ where $w = z$. Consequently, the problem reduces to the following lemma.

**Lemma 1.5.11.** *Let $F$ be a holomorphic function on a domain of holomorphy $D$ in $\mathbb{C}^N$. Suppose $1 \le k \le N$. If $F$ is identically equal to 0 on the slice of $D$ by the $(N-k)$-dimensional complex subspace where $z_1 = z_2 = \cdots = z_k = 0$ [in other words, if the function $F(0,\dots,0,z_{k+1},\dots,z_n)$ is identically equal to 0], then there are holomorphic functions $G_1$, …, $G_k$ on $D$ such that $F(z) = z_1G_1(z) + \cdots + z_kG_k(z)$ for all $z$ in $D$.*

To obtain Hefer's theorem from the lemma, set $N$ equal to $2n$ and $k$ equal to $n$. Take $D$ to be the image of the domain of holomorphy $\Omega \times \Omega$ under the invertible linear transformation that sends the variables $(w,z)$ to $(w-z,z)$. □

*Proof of the lemma.* That the conclusion of the lemma holds *locally* follows from considering the local power series expansion of $F$. The whole difficulty is to see that local solutions patch together to give a global solution. One way to achieve the patching is to use a proposition from last semester that every holomorphic function on a slice of a domain of holomorphy by an affine complex subspace of codimension 1 extends to be a holomorphic function on the domain. (The proof of that proposition uses the solvability of the $\overline{\partial}$-problem on domains of holomorphy.)

In this line of argument, the lemma is proved for all dimensions $N$ simultaneously by an induction on $k$. When $k = 1$, define $G_1$ as follows:

$$G_1(z_1, \dots, z_N) = \begin{cases} F(z_1, \dots, z_N)/z_1, & \text{if } z_1 \neq 0, \\ \partial F/\partial z_1, & \text{if } z_1 = 0. \end{cases}$$

Next consider a general value of $k$, supposing that the statement has been proved for smaller values (for every dimension $N$). Restrict $F$ to the $(N-1)$-dimensional slice of $D$ where $z_k = 0$. The induction hypothesis yields the existence of holomorphic functions $G_1$, $\dots$, $G_{k-1}$ on the slice such that

$$F(z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_N) = \sum_{j=1}^{k-1} z_j G_j(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_N).$$

The above-mentioned proposition from last semester shows that the $G_j$ extend from the slice to all of $D$ as holomorphic functions (which one might as well continue to call $G_j$).

Thus the function $F(z) - \sum_{j=1}^{k-1} z_j G_j(z)$ vanishes when $z_k = 0$. By the basis step of the induction, there is a holomorphic function $G_k$ on $D$ such that $F(z) - \sum_{j=1}^{k-1} z_j G_j(z) = z_k G_k(z)$ for all $z$ in $D$. Hence the conclusion of the lemma holds. □

As indicated above, a Hefer decomposition is feasible not only on domains of holomorphy but also on many non-pseudoconvex domains. On complete Reinhardt domains, for example, one can read off the Hefer decomposition from the (global) power series representation of $f$. More generally, if $\Omega$ has a schlicht envelope of holomorphy, then one can restrict a Hefer decomposition for the envelope to obtain a Hefer decomposition for $\Omega$. I do not know the answer to the following characterization question.

**Open problem 1.5.12.** For which (non-pseudoconvex) domains does the conclusion of Hefer's theorem hold?

## 1.6 The Bochner–Martinelli–Koppelman kernel

In his last publication [17], Koppelman made the influential observation that the machinery of Cauchy–Fantappiè kernels can be adapted to give integral representations not just for functions but also for differential forms. One can see the details worked out using determinant notation similar to Koppelman's notation in the book by Aĭzenberg and Dautov [2, §1]. Range's notation [29, Chapter IV, §1] uses the Hodge star operator. The notation used below is different from both of these.

**Definition 1.6.1.** In $\mathbb{C}^n$, let $\gamma$ denote the $(0,1)$-form $\sum_{j=1}^n (\overline{w}_j - \overline{z}_j)\, dw_j$. When $0 \le q \le n-1$, the Bochner–Martinelli–Koppelman kernel $U_q(w,z)$ equals

$$\frac{c_{n,q}}{|w-z|^{2n}}\, \gamma \wedge (\overline{\partial}_z \gamma)^q \wedge (\overline{\partial}_w \gamma)^{n-q-1},$$

where $c_{n,q} = \binom{n-1}{q}(-1)^{q(q-1)/2}/(2\pi i)^n$. Notice that the constant $c_{n,0}$ is related to but not equal to the constant $c_n$. (Technically, there should be another subscript on $U_q$ to indicate the dimension $n$, but omitting this second subscript simplifies the notation and should cause no confusion.) The kernel is understood as a *double differential form*, meaning that the differentials in the $z$ variables commute with the differentials in the $w$ variables. In other words, the kernel is a differential form in $w$ whose coefficients are differential forms in $z$. For convenience in writing general formulas, one defines both $U_{-1}$ and $U_n$ to be identically equal to 0.

**Exercise 1.6.2.** When $q = 0$, the kernel $U_q$ reduces to the Bochner–Martinelli kernel (1.15).

**Exercise 1.6.3.** Let $\beta$ denote $\gamma(w,z)/|w-z|^2$. Show that

$$U_q(w,z) = c_{n,q}\, \beta \wedge (\overline{\partial}_z \beta)^q \wedge (\overline{\partial}_w \beta)^{n-q-1}.$$

(Compare Exercise 1.5.2.)

**Theorem 1.6.4** (Koppelman)**.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with class $C^1$ boundary, and suppose $0 \le q \le n$. If $f$ is a $(0,q)$-form with coefficients of class $C^1$ on the closure of $\Omega$, then*

$$\int_{b\Omega} f(w) \wedge U_q(w,z) - \int_{\Omega} \overline{\partial}_w f(w) \wedge U_q(w,z) - \overline{\partial}_z \int_{\Omega} f(w) \wedge U_{q-1}(w,z) = \begin{cases} f(z), & z \in \Omega, \\ 0, & z \notin \mathsf{closure}(\Omega). \end{cases}$$

Before turning to the proof, one should check that all the terms in the equation make sense. Since $U_q(w,z)$ is a differential form of type $(n, n-q-1)$ in $w$, the differential form $f \wedge U_q$ has type $(n, n-1)$, and this is the right degree for integration over the boundary $b\Omega$.

The differential form $\overline{\partial} f \wedge U_q$ is then of type $(n, n)$ in $w$, and this is the right degree for integration over the domain $\Omega$. The differential form $U_{q-1}$ has type $(n, n-(q-1)-1)$ or $(n, n-q)$ in $w$, so $f \wedge U_{q-1}$ again has the right degree for integration over the domain $\Omega$.

The kernel $U_q$ has a singularity when $w = z$ of order $1/|w-z|^{2n-1}$ (the order of singularity is independent of $q$). The boundary integral does not see this singularity, for the integration variable $w$ is in the boundary, and the free variable $z$ is not in the boundary. The two interior integrals, however, do encounter the singularity. Both are absolutely convergent singular integrals (under the hypothesis that $f$ and $\overline{\partial} f$ are continuous on the closure of $\Omega$) because the order of the singularity is 1 unit less than the dimension of the underlying real Euclidean space.

A further argument is needed to see that $\int_\Omega f(w) \wedge U_{q-1}(w, z)$ is differentiable with respect to $z$, and this point is a central issue in the proof of the theorem. One cannot simply differentiate under the integral sign, because the derivative of $U_{q-1}$ has a singularity of order $2n$, and such a singularity is not integrable. To handle this difficulty, split $f$ as $\chi f + (1-\chi) f$, where $\chi$ is a smooth cut-off function that is identically equal to 1 in a neighborhood of the boundary and identically equal to 0 in a neighborhood of $z$. Then $\int_\Omega \chi(w) f(w) \wedge U_{q-1}(w, z)$ can be differentiated under the integral sign, because the singularity where $w = z$ is suppressed by the vanishing of $\chi(z)$. On the other hand, the remaining integral can be written as $\int_{\mathbb{C}^n} (1-\chi(w)) f(w) \wedge U_{q-1}(w, z)$, with the integration extended over the whole space, because $(1-\chi(w)) f(w)$ has compact support in $\Omega$. Since $U_{q-1}$ depends on the difference $w - z$, one can introduce a new integration variable equal to this difference. Then it becomes possible to differentiate under the integral sign with the differentiation acting on $f$.

*Proof of Theorem 1.6.4.* As in the proof of the Bochner–Martinelli integral representation, the idea is to apply the theorem of Stokes to convert the boundary integral into an integral over $\Omega$. When $w \neq z$, the kernel $U_q(w, z)$ is a smooth differential form in $w$ of type $(n, n-q-1)$, and $d_w(f(w) \wedge U_q(w, z)) = \overline{\partial}_w(f(w) \wedge U_q(w, z)) = \overline{\partial}_w f(w) \wedge U_q(w, z) + (-1)^q f(w) \wedge \overline{\partial}_w U_q(w, z)$. The following lemma is a key element in the proof.

**Lemma 1.6.5.** *The Bochner–Martinelli–Koppelman kernel $U_q$ satisfies the following identity:*

$$(-1)^q\, \overline{\partial}_w U_q(w, z) = \overline{\partial}_z U_{q-1}(w, z) \qquad \textit{when } w \neq z.$$

*Proof.* Using the notation of Exercise 1.6.3, observe that

$$\overline{\partial}_w U_q(w, z) = c_{n,q} \left( (\overline{\partial}_z \beta)^q \wedge (\overline{\partial}_w \beta)^{n-q} + (-1)^q q\, \beta \wedge (\overline{\partial}_z \beta)^{q-1} \wedge \overline{\partial}_w \overline{\partial}_z \beta \wedge (\overline{\partial}_w \beta)^{n-q-1} \right).$$

On the other hand,

$$\begin{aligned}&\overline{\partial}_z U_{q-1}(w,z)\\&\quad = c_{n,q-1}\left((\overline{\partial}_z\beta)^q \wedge (\overline{\partial}_w\beta)^{n-q} + (-1)^{q-1}(n-q)\,\beta\wedge(\overline{\partial}_z\beta)^{q-1}\wedge\overline{\partial}_z\overline{\partial}_w\beta\wedge(\overline{\partial}_w\beta)^{n-q-1}\right).\end{aligned}$$

A routine calculation shows that $qc_{n,q} = (-1)^{q-1}(n-q)c_{n,q-1}$. Working with double differential forms means that $\overline{\partial}_w\overline{\partial}_z = \overline{\partial}_z\overline{\partial}_w$, so it remains to show that $(\overline{\partial}_z\beta)^q \wedge (\overline{\partial}_w\beta)^{n-q} = 0$.

Introduce the notation $\sum_{j=1}^n \beta_j(w,z)\,dw_j$ for the $(1,0)$-form $\beta$. In other words, $\beta_j(w,z) = (\overline{w}_j - \overline{z}_j)/|w-z|^2$. Then $\sum_{j=1}^n \beta_j(w,z)(w_j - z_j) = 1$. Therefore

$$\sum_{j=1}^n \frac{\partial\beta_j}{\partial\overline{w}_k}(w,z)(w_j-z_j) = 0 \qquad\text{and}\qquad \sum_{j=1}^n \frac{\partial\beta_j}{\partial\overline{z}_k}(w,z)(w_j-z_j) = 0$$

for every index $k$. Now fix points $w$ and $z$, and let $M$ denote the complex subspace of $\mathbb{C}^n$ of dimension $n-1$ consisting of complex vectors $t$ such that $\sum_{j=1}^n t_j(w_j - z_j) = 0$. For each $k$, the component of $d\overline{z}_k$ in $\overline{\partial}_z\beta$ is a $(1,0)$-form in $w$ whose coefficients lie in $M$, and similarly for the component of $d\overline{w}_k$ in $\overline{\partial}_w\beta$. The wedge product of $n$ such $(1,0)$ forms in $w$ whose coefficients lie in an $(n-1)$-dimensional subspace of $\mathbb{C}^n$ must vanish. This establishes the claim that $(\overline{\partial}_z\beta)^q \wedge (\overline{\partial}_w\beta)^{n-q} = 0$ and thereby finishes the proof of the lemma. □

The proof of the lemma (compare [29, pp. 173–174]) works more generally for any Cauchy–Fantappiè form of order $q$.

The preceding discussion contains the complete proof of the theorem in the case that $z$ is outside the closure of $\Omega$. Indeed, in that case the kernel $U_q$ has no singularity inside $\Omega$, so Stokes's theorem implies that

$$\int_{b\Omega} f(w)\wedge U_q(w,z) = \int_\Omega \overline{\partial}_w f(w)\wedge U_q(w,z) + \int_\Omega (-1)^q f(w)\wedge\overline{\partial}_w U_q(w,z).$$

Applying the lemma and interchanging $\overline{\partial}_z$ with the integral in $w$ finishes the argument.

Moreover, essentially the same reasoning handles the case when the point $z$ is *inside* $\Omega$ if the coefficients of $f$ vanish at $z$. Indeed, one can apply Stokes's theorem to the region $\Omega\setminus B(z,\epsilon)$ and take the limit as $\epsilon\to 0$. The integral $\int_{bB(z,\epsilon)} f(w)\wedge U_q(w,z)$ tends to 0 because $U_q$ is of order $\epsilon^{-(2n-1)}$ on $bB(z,\epsilon)$, the surface area of $bB(z,\epsilon)$ is of order $\epsilon^{2n-1}$, and the continuous differential form $f(w)$ tends to 0 at $z$ by hypothesis. The integral $\int_{\Omega\setminus B(z,\epsilon)} \overline{\partial}_w f(w)\wedge U_q(w,z)$ tends to $\int_\Omega \overline{\partial}_w f(w)\wedge U_q(w,z)$ because $U_q$ has an integrable singularity when $w=z$. Finally, the term $\overline{\partial}_z\int_{\Omega\setminus B(z,\epsilon)} f(w)\wedge U_{q-1}(w,z)$ tends to the limit $\overline{\partial}_z\int_\Omega f(w)\wedge U_{q-1}(w,z)$ because the non-integrable singularity of order $2n$ of $\overline{\partial}_z U_{q-1}(w,z)$ is reduced one unit by the vanishing of the coefficients of $f(w)$ at $z$.

The general case can be handled by induction on $q$. The basis step ($q = 0$) is Theorem 1.3.4 for the Bochner–Martinelli kernel. Suppose, then, that the theorem has been established for forms of degree $q-1$, and let $f$ be a $(0,q)$-form.

Because Koppelman's formula is linear in $f$, it suffices to consider a differential form $f$ that consists of a single term $\phi(w)\,d\overline{w}_{j_1}\wedge\cdots\wedge d\overline{w}_{j_q}$. The Bochner–Martinelli–Koppelman kernel evidently is invariant under permutations of the coordinates, so there is no loss of generality in assuming that $f(w) = \phi(w)\,d\overline{w}_1\wedge\cdots\wedge d\overline{w}_q$. The observation above shows that the conclusion of the theorem holds for the differential form $(\phi(w)-\phi(z))\,d\overline{w}_1\wedge\cdots\wedge d\overline{w}_q$, so the problem reduces to considering the form $\phi(z)\,d\overline{w}_1\wedge\cdots\wedge d\overline{w}_q$. In other words, it is enough to prove Koppelman's formula for a form $f$ with *constant coefficients*: namely, the form $d\overline{w}_1\wedge\cdots\wedge d\overline{w}_q$.

Since $d\overline{w}_1\wedge\cdots\wedge d\overline{w}_q = d(\overline{w}_1\,d\overline{w}_2\wedge\cdots\wedge d\overline{w}_q)$, one has that

$$\int_{b\Omega} d\overline{w}_1\wedge\cdots\wedge d\overline{w}_q\wedge U_q(w,z) = \int_{b\Omega} \overline{w}_1\,d\overline{w}_2\wedge\cdots\wedge d\overline{w}_q\wedge(-1)^q\,d_wU_q(w,z).$$

By Lemma 1.6.5, the latter integral equals $\overline{\partial}_z\int_{b\Omega}\overline{w}_1\,d\overline{w}_2\wedge\cdots\wedge d\overline{w}_q\wedge U_{q-1}(w,z)$. The induction hypothesis implies that this expression equals

$$\overline{\partial}_z\left(\int_{\Omega} d\overline{w}_1\wedge d\overline{w}_2\wedge\cdots\wedge d\overline{w}_q\wedge U_{q-1}(w,z) + \overline{\partial}_z(\text{irrelevant}) + \overline{z}_1\,d\overline{z}_2\wedge\cdots\wedge d\overline{z}_q\right).$$

For the constant-coefficient form $f$ under consideration (which in particular is $\overline{\partial}$-closed), this expression is equal to $\int_\Omega \overline{\partial}_w f(w)\wedge U_q(w,z) + \overline{\partial}_z\int_\Omega f(w)\wedge U_{q-1}(w,z) + f(z)$. That completes the induction step and thus finishes the proof of the theorem. □

A noteworthy consequence of Koppelman's theorem is that for $(0,n)$-forms (forms of top degree), one can solve the $\overline{\partial}$-equation immediately by an integral formula. In particular, no geometric hypothesis is needed on the domain for solvability of the $\overline{\partial}$-problem in top degree.

Koppelman's theorem also demonstrates that every compactly supported, $\overline{\partial}$-closed $(0,q)$-form in $\mathbb{C}^n$ (where $1\le q\le n$) is $\overline{\partial}$-exact. (Apply the theorem on a large ball containing the support, in which case the boundary integral disappears.) When $q = n$, the solution does not necessarily have compact support, even if the data has compact support. (The top-dimensional case is analogous to case when the dimension $n = 1$.) When $1\le q\le n-1$, however, there is a solution with compact support, but this proposition is harder to prove when $q > 1$ than when $q = 1$.

### 1.6.1 Solving $\overline{\partial}$ on convex domains

Combining the Bochner–Martinelli–Koppelman integral representation with the ideas of Cauchy–Fantappiè forms leads to an explicit integral formula for solving the $\overline{\partial}$-problem on a bounded convex domain. Recall from section 1.5.1 that if $\rho$ is the defining function of a convex domain $\Omega$, then $\sum_{j=1}^{n} \frac{\partial\rho}{\partial w_j}(w)(w_j - z_j) \neq 0$ when $w \in b\Omega$ and $z \in \Omega$. By analogy with the definition of the Bochner–Martinelli–Koppelman kernel, one can construct on convex domains a Cauchy–Fantappiè-Koppelman kernel for differential forms as follows.

Let $\widetilde{\gamma}(w)$ denote the $(0,1)$-form $\sum_{j=1}^{n} \frac{\partial\rho}{\partial w_j}(w)\, dw_j$, and let $\widetilde{\beta}(w,z)$ denote the quotient

$$\widetilde{\gamma}(w) \Big/ \sum_{j=1}^{n} \frac{\partial\rho}{\partial w_j}(w)(w_j - z_j)$$

for $w$ in $b\Omega$ and $z \in \Omega$. Let $\widetilde{U}_q(w,z)$ denote the kernel $c_{n,q}\,\widetilde{\beta} \wedge (\overline{\partial}_z\widetilde{\beta})^q \wedge (\overline{\partial}_w\widetilde{\beta})^{n-q-1}$, as in Exercise 1.6.3. Since the coefficients of $\widetilde{\beta}$ depend holomorphically on $z$, one has that $\widetilde{U}_q(w,z) = 0$ when $q \geq 1$. One cannot, however, simply replace $U_q$ in Theorem 1.6.4 with $\widetilde{U}_q$, because $\widetilde{U}_q$ is not well defined when $w$ is inside $\Omega$ (since the denominator can be 0). Nonetheless, one can use $\widetilde{U}_q$ to construct an integral solution operator for $\overline{\partial}$ by using the following lemma, which actually holds more generally for the difference of every pair of Cauchy–Fantappiè-Koppelman forms.

**Lemma 1.6.6** (Koppelman)**.** *When $w$ is in the boundary of $\Omega$ and $z$ is in the interior of $\Omega$, the difference $U_q(w,z) - \widetilde{U}_q(w,z)$ can be written as the sum of two (explicit) differential forms, one of which is $\overline{\partial}_w$-exact and the other of which is $\overline{\partial}_z$-exact.*

Assuming the lemma for the moment, subtract $\int_{b\Omega} f(w) \wedge \widetilde{U}_q(w,z)$ (which equals 0 when $q \geq 1$) in the formula of Theorem 1.6.4. Using the lemma to introduce differential forms $\Phi$ and $\Psi$ such that $U_q(w,z) - \widetilde{U}_q(w,z) = \overline{\partial}_w\Phi(w,z) + \overline{\partial}_z\Psi(w,z)$, one then has when $q \geq 1$ that

$$\int_{b\Omega} f(w) \wedge \overline{\partial}_w\Phi(w,z) + \overline{\partial}_z \int_{b\Omega} f(w) \wedge \Psi(w,z) \\ - \int_{\Omega} \overline{\partial}_w f(w) \wedge U_q(w,z) - \overline{\partial}_z \int_{\Omega} f(w) \wedge U_{q-1}(w,z) = f(z), \qquad z \in \Omega.$$

When $f$ is a $\overline{\partial}$-closed $(0,q)$-form, and $q \geq 1$, the preceding equation reduces to the statement that

$$f(z) = \overline{\partial}_z\biggl(\int_{b\Omega} f(w) \wedge \Psi(w,z) - \int_{\Omega} f(w) \wedge U_{q-1}(w,z)\biggr).$$

Thus one has an explicit integral solution operator for the $\overline{\partial}$-equation on bounded convex domains.

From the explicit formula, one can deduce regularity properties of the solution. As will be shown below, the coefficients of the form $\Psi(w,z)$ depend on $\beta$ and $\widetilde{\beta}$ and their first derivatives. Consequently, $\Psi(w,z)$ is class $C^\infty$ in $z$ inside $\Omega$ uniformly for $w$ in $b\Omega$. Therefore the first term, $\int_{b\Omega} f(w)\wedge\Psi(w,z)$, belongs to class $C^\infty(\Omega)$ if $f$ is continuous on $b\Omega$. For the second term, $\int_\Omega f(w)\wedge U_{q-1}(w,z)$, consider separately the cases of $f$ supported away from $z$ and $f$ with compact support. In the first case, the integral depends smoothly on $z$ since $U_{q-1}(w,z)$ is smooth when $z\neq w$. In the second case, use that $U_{q-1}(w,z)$ depends only on the difference $w-z$; after introducing a new integration variable equal to $w-z$, one can differentiate under the integral sign to see that the integral is as smooth as $f$ is. In conclusion, if $f$ is a $\overline{\partial}$-closed $(0,q)$-form (where $q\geq 1$) of class $C^k$ (where $k\geq 1$) on the closure of a bounded convex domain $\Omega$, then the above integral solution operator produces a $(0,q-1)$-form $u$ of class $C^k(\Omega)$ such that $\overline{\partial}u=f$. (Compare [29, pp. 172–176].)

*Proof of Lemma 1.6.6.* Convexity is not needed in the proof of the lemma. The proof uses only the property that the coefficients $\widetilde{\beta}_j$ of the $(1,0)$-form $\widetilde{\beta}$ have been chosen such that $\sum_{j=1}^n \widetilde{\beta}_j(w,z)(w_j-z_j)=1$ (and similarly for $\beta$). In other words, the lemma (like Lemma 1.6.5) is a general combinatorial fact about Cauchy–Fantappiè-Koppelman forms.

The idea is to make a homotopy between $U_q$ and $\widetilde{U}_q$. First, let $\alpha(w,z,t)$ denote the sum $t\beta(w,z)+(1-t)\widetilde{\beta}(w,z)$. Next, define a differential form $V_q$ such that

$$V_q(w,z,t)=c_{n,q}\,\alpha\wedge(\overline{\partial}_z\alpha)^q\wedge(\overline{\partial}_w\alpha)^{n-q-1}.$$

Finally, observe that

$$U_q(w,z)-\widetilde{U}_q(w,z)=\int_0^1 d_tV_q(w,z,t).$$

(In the differential form $d_tV_q$, the three variables $w$, $z$, and $t$ are viewed as commuting with each other.) The proof of the lemma now reduces to showing that $d_tV_q$ is the sum of a $\overline{\partial}_w$-exact form and a $\overline{\partial}_z$-exact form.

The technical tool in the argument is the same one as in the proof of Lemma 1.6.5. Namely, the coefficients $\alpha_j$ of the $(1,0)$-form $\alpha$ have the property that $\sum_{j=1}^n \alpha_j(w,z,t)=1$, so $d_t\alpha$ equals $dt$ times a $(1,0)$-form in $w$ whose coefficients lie in the $(n-1)$-dimensional subspace $M$ of $\mathbb{C}^n$ consisting of complex vectors $t$ such that $\sum_{j=1}^n t_j(w_j-z_j)=0$. Similarly, for each $k$ the $d\overline{z}_k$ component of $\overline{\partial}_z\alpha$ is a $(1,0)$-form in $w$ whose coefficients lie in $M$; and for each $k$ the $d\overline{w}_k$ component of $\overline{\partial}_w\alpha$ is a $(1,0)$-form in $w$ whose coefficients lie in $M$. The wedge product of $n$ forms, each of which is one of $d_t\alpha$, $\overline{\partial}_z\alpha$, and $\overline{\partial}_w\alpha$, must vanish, since there cannot exist $n$ linearly independent $(1,0)$-forms in $w$ whose coefficients lie in an $(n-1)$-dimensional subspace of $\mathbb{C}^n$.

In view of the preceding observation,

$$\begin{aligned}d_tV_q=c_{n,q}\big(0+q\,\alpha\wedge d_t\overline{\partial}_z\alpha\wedge(\overline{\partial}_z\alpha)^{q-1}\wedge(\overline{\partial}_w\alpha)^{n-q-1}&\\ +(n-q-1)\,\alpha\wedge(\overline{\partial}_z\alpha)^q\wedge d_t\overline{\partial}_w\alpha\wedge(\overline{\partial}_w\alpha)^{n-q-2}\big)&.\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{\partial}_z \left(\alpha \wedge d_t\alpha \wedge (\overline{\partial}_z\alpha)^{q-1} \wedge (\overline{\partial}_w\alpha)^{n-q-1}\right) &= \alpha \wedge \overline{\partial}_z d_t\alpha \wedge (\overline{\partial}_z\alpha)^{q-1} \wedge (\overline{\partial}_w\alpha)^{n-q-1} \\ &\quad + (-1)^{q-1}(n-q-1)\, \alpha \wedge d_t\alpha \wedge (\overline{\partial}_z\alpha)^{q-1} \wedge \overline{\partial}_z\overline{\partial}_w\alpha \wedge (\overline{\partial}_w\alpha)^{n-q-2},\end{aligned}$$

and

$$\begin{aligned}&\overline{\partial}_w \left(\alpha \wedge d_t\alpha \wedge (\overline{\partial}_z\alpha)^{q} \wedge (\overline{\partial}_w\alpha)^{n-q-2}\right) \\ &\quad = -\alpha \wedge \overline{\partial}_w d_t\alpha \wedge (\overline{\partial}_z\alpha)^{q} \wedge (\overline{\partial}_w\alpha)^{n-q-2} + q\, \alpha \wedge d_t\alpha \wedge \overline{\partial}_w\overline{\partial}_z\alpha \wedge (\overline{\partial}_z\alpha)^{q-1} \wedge (\overline{\partial}_w\alpha)^{n-q-2}.\end{aligned}$$

Comparing these three equations shows that $d_tV_q = \overline{\partial}_w\phi + \overline{\partial}_z\psi$, where

$$\phi = -(n-q-1)c_{n,q}\, \alpha \wedge d_t\alpha \wedge (\overline{\partial}_z\alpha)^{q} \wedge (\overline{\partial}_w\alpha)^{n-q-2}$$

and

$$\psi = qc_{n,q}\, \alpha \wedge d_t\alpha \wedge (\overline{\partial}_z\alpha)^{q-1} \wedge (\overline{\partial}_w\alpha)^{n-q-1}.$$

Consequently, $U_q - \widetilde{U}_q = \overline{\partial}_w\Phi + \overline{\partial}_z\Psi$, where $\Phi = \int_0^1 \phi$ and $\Psi = \int_0^1 \psi$. $\square$

Knowing the solvability of the $\overline{\partial}$-equation on bounded convex domains leads to a proof of Lemma 1.3.38 (a point that was left open on page 32). Although that lemma stated solvability of the $\overline{\partial}$-equation on $(0,1)$-forms in class $C^1(\mathbb{C}^n)$, the following more general statement is no more difficult to prove.

**Theorem 1.6.7.** *If $G$ is a $\overline{\partial}$-closed $(0,q)$-form (where $q \geq 1$) of class $C^k$ (where $k \geq 1$) on the whole space $\mathbb{C}^n$, then there exists a $(0,q-1)$-form $u$ of class $C^k$ on $\mathbb{C}^n$ such that $\overline{\partial}u = G$.*

*Proof.* Exhaust $\mathbb{C}^n$ by an increasing sequence of nested, concentric, closed balls $B_j$. In view of the discussion above, one can find a form $u$ of class $C^k(\mathbb{C}^n)$ such that $\overline{\partial}u = G$ in a convex neighborhood of $B_j$. The difficulty is to splice together these solutions for different values of $j$ to get a global solution.

When $q = 1$, one can do the splicing by the "Mittag-Leffler trick" that is familiar from the theory of meromorphic functions of one complex variable. In this case, $u$ is a function (a $(0,0)$-form). The idea is to construct recursively a sequence of functions $u_j$ of class $C^k(\mathbb{C}^n)$ such that $\overline{\partial}u_j = G$ in a convex neighborhood of $B_j$ and $|u_{j+1} - u_j| < 2^{-j}$ on $B_j$. Then the telescoping sum $u_1 + (u_2 - u_1) + (u_3 - u_2) + \cdots$ will converge, uniformly on compact sets, to a function $u$ such that $\overline{\partial}u = G$ in $\mathbb{C}^n$. Supposing that a certain $u_j$ has already been constructed, to construct $u_{j+1}$ one first finds $u$ such that $\overline{\partial}u = G$ in a neighborhood of $B_{j+1}$. Then $\overline{\partial}(u - u_j) = 0$ in a neighborhood of $B_j$, so there exists a holomorphic polynomial $v$ such that $|u - u_j - v| < 2^{-j}$ in a neighborhood of $B_j$. (For example, $v$ could be a partial sum of the Taylor series of $u - u_j$.) The function $u - v$ will serve for $u_{j+1}$.

When $q > 1$, the approximation argument is unnecessary, for one can construct the sequence of forms $u_j$ to have the stability property that $u_{j+1} = u_j$ on $B_j$. The existence of the limit $\lim_{j\to\infty} u_j$ is then trivial. To construct $u_{j+1}$ (supposing that $u_j$ has already been constructed), first find $u$ such that $\overline{\partial} u = G$ in a neighborhood of $B_{j+1}$. Then $\overline{\partial}(u - u_j) = 0$ in a neighborhood of $B_j$, so there exists a form $v$ of class $C^k(\mathbb{C}^n)$ such that $\overline{\partial} v = u - u_j$ in a neighborhood of $B_j$. The form $u - \overline{\partial} v$ will serve for $u_{j+1}$. □

The method of proof of Theorem 1.6.7 applies more generally to show that the class of domains on which the $\overline{\partial}$-equation is solvable (on forms of all degrees) is closed under increasing unions. (Compare [10, proof of Theorem 2.7.8].)

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