# Who Was Stefan Bergman?

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In accordance with the new California state truth-in-lecturing law, I am constrained to begin with a disclaimer. Contrary to standard practice, I am going to give a lecture that everybody can understand. Please don't tell Bill Thurston, or he may revoke my invitation to come back in the spring.

Last year at a teaching workshop, I had a conversion experience and became a born-again Socratic. I have not spent a full class period lecturing ever since then, so I feel that I need an excuse for lecturing today. The excuse for this talk is that Emil Straube and I received the Bergman Prize this year. Since there is no award ceremony for this prize, there is no public opportunity to honor the memory and laud the achievements of Stefan Bergman. I would like to do so today.

I never met Stefan Bergman. He died in the summer of 1977, when I had just completed my first year of graduate studies and did not even know what several complex variables is all about. So what I am going to tell you I know from secondary and tertiary sources.

There is a story about Stefan Bergman to the effect that Zeev Nehari once showed him a manuscript, and Bergman said: "That's a nice paper. Let's make it a joint paper, and I'll write the next one."

I have heard this story from two different sources, so it might even be true, but you should not interpret the story as reflecting poorly on Bergman. In fact, from what I have heard, Bergman was extremely generous with students and young colleagues, both with time and even with money. Such generosity is a trait one sometimes sees in people who have themselves lived on the edge.

The conclusion you should draw from the story is that Bergman was a bit eccentric, which is a defining characteristic of a true mathematician. Some people, even some people in this room, might go further and say that anyone who devotes single-minded attention to an abstruse subject with few tangible rewards must be a little bit crazy. I think you will agree that the Bergman in the picture looks like a mathematician.

I am going to tell you a little about Bergman's life and mathematics, but you must keep in mind that he is a man from a different era. When he was born near the end of the last century, Weierstrass was still alive, the Hilbert problems had not been stated, Poincaré was at the height of his powers, Einstein's special theory of relativity was a decade ahead. When Bergman was at university, the subject of functional analysis did not exist.

Bergman was born in 1895 in Częstochowa, the site of a famous Catholic shrine and now a provincial capital in Poland. At the time, Poland did not exist as a political entity, having been partitioned among Austria, Germany, and Russia, and so Bergman grew up officially in Russia. At university, he studied engineering, and so it was natural for him to move in 1921 to the Institute of Applied Mathematics that von Mises had just established in Berlin. It was in Berlin that Bergman began work on his most famous contribution to pure mathematics, the Bergman kernel function.

Let me say for the record what the Bergman kernel function is in the simplest possible setting of the unit disk in the complex plane. If we are going to do anything in complex analysis, we need Cauchy's integral formula for analytic functions in the unit disk:

$$f(w) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z-w}\, dz.$$

How many of you have recently taught multi-variable calculus? If you do it the same way as at my university, there is too much material in the syllabus, and you get to the theorems of Green, Gauss, and Stokes on the last day of class (and you then make that a major part of the final examination).

I don't know how interesting the applications we give our students are, but if you apply Green's theorem to Cauchy's formula to convert the line integral to an area integral:

$$f(w) = \frac{1}{\pi} \int_{|z|<1} \frac{f(z)}{(1-w\bar{z})^2}\, dx\, dy,$$

then the kernel in this integral is the *Bergman kernel function*

$$K(w,z) = \frac{1}{\pi(1-w\bar{z})^2}$$

for the unit disk. It is a reproducing kernel function: integrating against it recovers the values of an analytic function.

So in some sense, we could claim that Cauchy knew about the Bergman kernel. What makes the Bergman kernel more interesting than just an explicit integral representation formula is that there are two other ways to view the Bergman kernel. One way is as an infinite series in terms of an orthonormal basis $\{\phi_j\}$ for the square-integrable analytic functions:

$$K(w, z) = \sum_j \phi_j(w)\overline{\phi}_j(z).$$

Another way to view the Bergman kernel function $K(w, z)$ is as the solution of the extremal problem of maximizing $f(z)$ (for a specified point $z$) subject to the constraint $f(z) \geq \int |f|^2$. The extremal function is $K(w, z)$.

Why do we care about the Bergman kernel function? I will mention four reasons.

The first reason has to do with developments later in Bergman's career, when he applied kernel methods to the study of partial differential equations. You are familiar with the idea of a Green function or a fundamental solution for a partial differential equation. Bergman proposed that there are three questions to ask about solutions of partial differential equations:

1. How do they change with the boundary conditions?
2. How do they change with the coefficients of the equation? (stability)
3. How do they change with the domain?

His idea was to develop a thorough understanding of kernel functions for simple equations and simple domains, and then apply stability and perturbation results to understand general equations on general domains.

The other reasons are connected with function theory. One basic tool in complex function theory is the Schwarz lemma, which can be formulated as the statement that of all analytic functions in a disk that are zero at the center and have derivative one at the center, the ones with the smallest supremum are linear. The Schwarz lemma has no obvious analogue in general domains. Bergman's point of view was that the Schwarz lemma solves an extremal problem for analytic functions: so does the Bergman kernel function. More generally, one should study other extremal problems for analytic functions in several complex variables, and Bergman developed such a theory. This old idea has lately been revived fruitfully by Steve Krantz's recent student Jiye Yu.

Another basic result of one complex variable is the Riemann mapping theorem, which says that a bounded domain that is topologically equivalent to the unit disk is also conformally equivalent to the unit disk. There is no immediate analogue in higher dimensions. However, the Riemann mapping function $f$ can be written in terms of the Bergman kernel function via

$$f'(z) = K(z,a)\sqrt{\frac{\pi}{K(a,a)}}.$$

Studying the Bergman kernel function is therefore a natural way to try to generalize the Riemann mapping theorem to higher dimensions.

The geometers in the audience know and love the Poincaré metric in the unit disk, which is conformally invariant and which makes the disk a model for hyperbolic geometry. Bergman had the idea of introducing a holomorphically invariant metric in higher dimensions via

$$g_{jk}(z) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K(z,z).$$

This *Bergman metric* has turned out to be an important tool in complex function theory and geometry.

There is a story about how Bergman discovered the kernel function by mistake. In Berlin, Bergman attended the seminar of Erhard Schmidt (known to all students of linear algebra from the Gram-Schmidt procedure). Perhaps Schmidt had been lecturing on Fourier series; at any rate, Bergman either was assigned or took upon himself the project of working out the theory of expanding functions in terms of a general system of orthogonal functions. Maybe Schmidt had been using the letter $z$ for the variable, or maybe Bergman's German was imperfect, because Bergman got confused and looked at complex functions instead of real functions. He discovered that for functions of a complex variable, the convergence of the series was much better than for real functions: uniform on compact sets, and he ended up writing a dissertation about orthogonal systems of analytic functions and the kernel function. I heard the late Lipman Bers tell this story three times, and it was different each time, but I believe that there is a kernel of truth to it.

Bergman managed to get a position in Berlin, but when Hitler came to power in 1933, Berlin was no place for a Polish intellectual. Bergman went to

Russia. This may seem a strange choice to us, but you must remember that Bergman grew up reciting classic Russian poetry. Also, it was still possible to believe in 1933 that the Soviet Union was headed toward a Communist utopia, rather than a half-century of Communist dictatorship. But in 1937, Stalin began to purge scientists, and Russia was no place for a Polish intellectual. Bergman went to Paris. In 1939 the war came, and Paris was no place for a Polish intellectual. Bergman came to the United States, and he ended up spending the war years at Brown University.

In the current difficult job market, we worry about finding positions for our young colleagues who speak English with an accent. Today, the consequence for those who fail to find jobs may be that they leave mathematics for statistics or computer science. We should not forget that there was a time when getting a position in this country was truly a matter of life or death. Some of you may have heard Lipman Bers speak far more poignantly than I can about the gratitude which the mathematical refugees of the 1930's felt for the warm hospitality they received from their American colleagues. I would like to think that this spirit of hospitality is still alive today.

After the war, Bergman joined his teacher von Mises at Harvard. When von Mises died, Bergman went to Stanford, where he spent the last quarter century of his life. I believe that at Stanford, Bergman found peace to do his work and to enjoy life. Certainly the second picture shows a more contented-looking Bergman.

Bergman's wife survived him by ten years, and her will established a trust with the Wells Fargo Bank to benefit researchers in real and complex analysis and partial differential equations with special attention to Bergman's methods. The bank's understanding of mathematics is limited to interest rates and present value (it does not know a PDE from a UFD), so it delegated the selection of beneficiaries to the American Mathematical Society, which selects the prize winners.

As I said at the beginning, the reason I am giving this talk is that Emil Straube and I were selected as the co-winners this year. Several curious colleagues have asked the amount of the prize, and I can honestly say that I do not know: it depends on how the trust's investments are doing, and we have not seen a check yet. We hope that the trust has not been investing in the Mexican stock market. But in previous years the prize has been announced in the AMS *Notices* as being approximately $20,000.

In the remainder of the talk, I would like to describe some of the mathematics that has been recognized by the Bergman Prize. The award winners to date are the following:

**David Catlin** (1989) (*Princeton*, Chicago, *Princeton*, Purdue)

**Steve Bell and Ewa Ligocka** (1991) (MIT, *Princeton*, Purdue; Warsaw, Polish Academy of Sciences)

**Charles Fefferman** (1992) (*Princeton*, Chicago, *Princeton*)

**Yum-Tong Siu** (1993) (*Princeton*, Purdue, Notre Dame, Yale, Stanford, Harvard)

**John Erik Fornæss** (1994) (Washington, *Princeton*, Michigan)

**Harold Boas and Emil Straube** (1995) (MIT, Columbia, TAMU; ETH, Indiana, Pittsburgh, TAMU)

I have listed the winners' doctoral institutions and subsequent teaching positions, because there is an interesting pattern. All recipients prior to this year have had a Princeton connection. It seems that the award committee finally recognized that there is significant mathematics being done to the west of Fine Hall.

It would take all day to describe the accomplishments of any one of the prize winners, so I will just highlight a few selected results. One of Bergman's interests was to try to visualize domains in two-dimensional complex space. If Bergman had had modern-day computers, no doubt he would have been making pictures of domains like the Diederich-Fornæss worm domain defined by $|w + e^{i \log |z|^2}|^2 - 1 + \lambda(|z|^{-2} - 1) + \lambda(|z|^2 - r^2) < 0$, where $\lambda$ is a smooth function that is zero on the negative axis and very convex on the positive axis.

Fefferman was cited for several contributions, but the big one that everybody remembers is his theorem from 1974 about biholomorphic mappings. Since there is no Riemann mapping theorem in several complex variables, people have been trying to extend related theorems from one variable to many. For example, one could ask about the boundary behavior of biholomorphic maps (that is, holomorphic maps with holomorphic inverses). One important theorem from one complex variable is Carathéodory's theorem saying that the Riemann map from a Jordan domain (one bounded by a simple closed Jordan curve) onto the unit disk extends to a homeomorphism of the closures. There is a simple example (although it was not discovered until years after Fefferman's work) showing that the analogous statement in higher dimensions is false. But if one assumes that the boundaries are smooth, then something can be said.

To state Fefferman's theorem, I need to define one term: the notion of pseudoconvexity. Everybody knows what a convex domain is: the line segment joining each pair of points of the domain stays inside the domain. A domain is *strongly convex* if it is like the region in the plane above the graph of $y = x^2$, but not like the region above $y = x^4$. Now convexity is not a natural property in complex analysis, because it is not invariant under holomorphic coordinate changes. A domain in complex space is *strongly pseudoconvex* if near each boundary point, it can be made strongly convex by a local holomorphic change of coordinates. A strongly convex domain is therefore strongly pseudoconvex, but not conversely. An interesting example of a non-convex, but strongly pseudoconvex domain is a barbell.

It is a bit unfortunate that the basic objects in complex function theory have unwieldy names, like pseudoconvex and plurisubharmonic. Bergman coined many new terms as he investigated new areas in several complex variables, and fortunately most of them have not survived. He wrote one paper on quasi-pseudo-conformal mappings!

**Theorem (Fefferman).** *Suppose $f : D \to G$ is a biholomorphic mapping between bounded strongly pseudoconvex domains in $\mathbf{C}^n$ with boundaries of class $C^\infty$. Then $f$ extends to a diffeomorphism of the closures.*

Fefferman's proof is based on the following idea. If we equip the domains with their invariant Bergman metrics, then the mapping $f$ is an isometry. To get the extension to the boundary, we walk out toward the boundary along geodesics, and hope something nice happens. To understand the asymptotics of the geodesics near the boundary, we need precise information about the behavior of the Bergman kernel function near the boundary. This is obtained by Bergman's method of comparison domains, using that a strongly pseudoconvex domain is locally well approximated by a ball. This paper is one that gets the description "tour de force."

The work for which Bell and Ligocka were cited is a new, simple proof and generalization of Fefferman's theorem based on new ideas. They found a function-theoretic connection between the Bergman kernel function and biholomorphic mappings. Their theorem is most easily stated in terms of the *Bergman projection*, which is the mapping that takes a nonholomorphic function into its nearest neighbor in the holomorphic functions.

**Theorem (Bell and Ligocka).** *Suppose $f : D \to G$ is a biholomorphic mapping between bounded domains in $\mathbf{C}^n$ with boundaries of class $C^\infty$, and suppose the Bergman projections preserve the space of functions that are $C^\infty$ on the closure. Then $f$ extends to a diffeomorphism of the closures.*

A new element in their proof is the use of Bergman representative coordinates, in which the mapping becomes linear. These special coordinates are cooked up from the Bergman kernel function, and so one needs to know about smoothness properties of the kernel function at the boundary.

Thus Fefferman's theorem reduces to understanding the Bergman projection operator. The most successful approach to this so far has been via partial differential equations. The basic PDE's in complex analysis are the Cauchy-Riemann equations, the holomorphic functions being the ones annihilated by the Cauchy-Riemann equations. If one studies the inhomogeneous Cauchy-Riemann equations, then clearly the solution is not unique: it can always be changed by addition of a homogeneous solution, that is, by a holomorphic function. To single out a unique solution, a natural procedure is to subtract off the projection onto the holomorphic functions, that is, to take the solution orthogonal to the holomorphic functions. This, then, is the connection between the Bergman projection and the Cauchy-Riemann equations. This canonical solution orthogonal to the holomorphic functions is called the (Kohn) solution to the $\overline{\partial}$-Neumann problem.

The hypothesis in the theorem of Bell and Ligocka reduces, in effect, to being able to solve the $\overline{\partial}$-Neumann problem with preservation of boundary smoothness. It turns out that the $\overline{\partial}$-Neumann problem is not an elliptic boundary value problem: the operator is elliptic (in the interior), but the boundary conditions are not the standard ones for elliptic theory. The $\overline{\partial}$-Neumann problem is the prototype of what are called noncoercive boundary value problems. Joe Kohn proved that the $\overline{\partial}$-Neumann problem is what is called *subelliptic* on strongly pseudoconvex domains. It is not as well-behaved as an elliptic problem, but it is good enough to have the boundary smoothness required in the Bell-Ligocka approach to Fefferman's theorem.

**Theorem (Catlin).** *A necessary and sufficient condition for the $\overline{\partial}$-Neumann problem to be subelliptic on a (weakly) pseudoconvex domain is that the boundary have bounded order of contact with complex varieties.*

This important result, in particular, extends Fefferman's theorem to a wide class of domains. You are supposed to be keeping in mind as examples complex analogues of $y = x^{2k}$. Catlin's work brought in some highly original ideas concerning the boundary geometry of domains in complex space.

Thus, we can prove a mapping theorem if we understand the Bergman projection, and we understand the Bergman projection if we understand the $\overline{\partial}$-Neumann problem. Emil Straube and I developed a theory leading to the following result.

**Theorem (Boas and Straube).** *Fefferman's mapping theorem holds for all (weakly) convex domains.*

*More generally, the $\overline{\partial}$-Neumann problem is regular in both $C^\infty$ and each Sobolev space $W^k$ for domains that are defined by plurisubharmonic functions.*

In view of all the positive results, it had been suspected for a long time that the $\overline{\partial}$-Neumann problem might be $C^\infty$ regular in every smooth bounded pseudoconvex domain. This has been recently disproved by Michael Christ, as we will hear tomorrow morning. He has proved the following theorem.

**Theorem (Michael Christ, 1995).** *The $\overline{\partial}$-Neumann problem is not $C^\infty$ regular in the Diederich-Fornæss worm domain.*

In closing, I should mention that I have only discussed some of Bergman's contributions in *pure* mathematics. Actually, it would not be much exaggeration to call Bergman an applied mathematician. For example, in his book with Schiffer, he wrote the following: "We are dealing with a field of applied mathematics where the interaction between mathematics and physics is highly stimulating and fruitful for both sciences." Bergman had a strong interest in applied mathematics and worked on fluid dynamics, for example. He wrote one paper titled "Punch-card machine methods applied to the solution of the torsion problem" (Quart. Appl. Math. 5 (1943), 69–81). Here is what he wrote in the preface to his book in 1950: "The actual computation of a complete orthonormal system is a rather time-consuming procedure; but with the aid of modern computational technique, these processes come well within the range of practical application."

Finally, one more Bergman joke. Bergman once said, "I speak six languages, but English is my bestest." If you want to hear more Bergman jokes, take Steve Krantz out to dinner after his talk this afternoon and get him drunk.